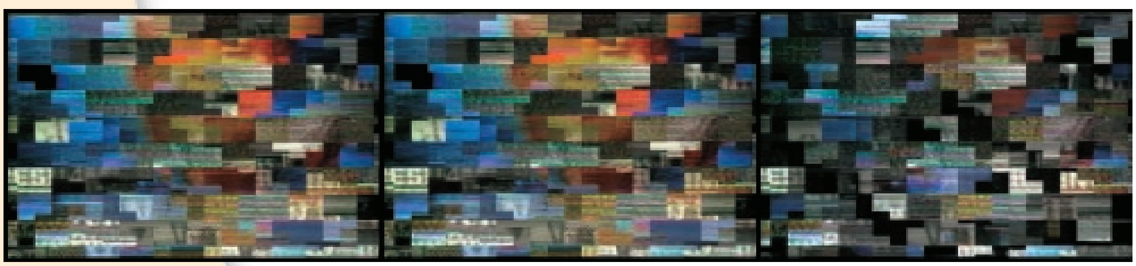
CHAPTER

1 Patterns and Recursion



To create the video piece *Residual Light,* experimental video artist Anthony Discenza (American, b 1967) recorded 3 hours of commercial television by filming the TV screen while continuously channel surfing. This 3-hour sample was then compressed in stages by recording and re-recording the material on analog and digital tape while controlling the speed of the playback. Through this recursive process, the original 3 hours was gradually reduced to just 3 minutes. This 3-minute sequence was then slowed back down, resulting in a 20-minute loop.

OBJECTIVES

In this chapter you will

- Recognize and visualize mathematical patterns called sequences
- Write recursive definitions for sequences
- Display sequences with graphs
- Investigate what happens to sequences in the long run

LESSON

1.1 Recursively Defined Sequences

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For every pattern that appears, a mathematician feels he ought to know why it appears.

W. W. SAWYER

Look around! You are surrounded by patterns and influenced by how you perceive them. You have learned to recognize visual patterns in floor tiles, window panes, tree leaves, and flower petals. In every discipline, people discover, observe, re-create, explain, generalize, and use patterns. Artists and architects use patterns that are attractive or practical. Scientists and manufacturing engineers follow patterns and predictable processes that ensure quality, accuracy, and uniformity. Mathematicians frequently encounter patterns in numbers and shapes.

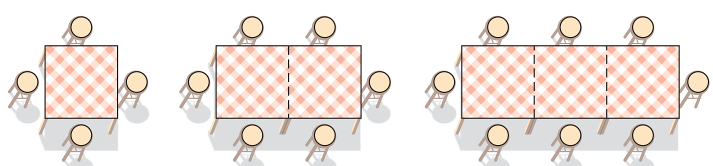
The arches in the Santa Maria Novella cathedral in Florence, Italy, show an artistic use of repeated patterns.

Scientists use patterns and repetition to conduct experiments, gather data, and analyze results.

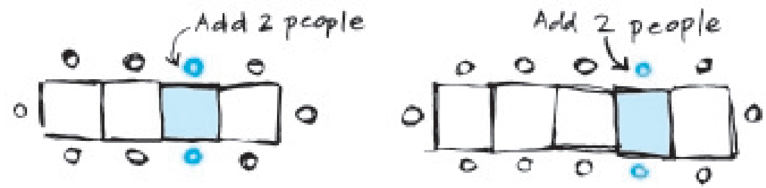
You can discover and explain many mathematical patterns by thinking about recursion. **Recursion** is a process in which each step of a pattern is dependent on the step or steps that come before it. It is often easy to define a pattern recursively, and a recursive definition reveals a lot about the properties of the pattern.

EXAMPLE A

A square table seats 4 people. Two square tables pushed together seat 6 people. Three tables pushed together seat 8 people. How many people can sit at 10 tables arranged in a straight line? How many tables are needed to seat 32 people? Write a recursive definition to find the number of people who can sit at any linear arrangement of square tables.



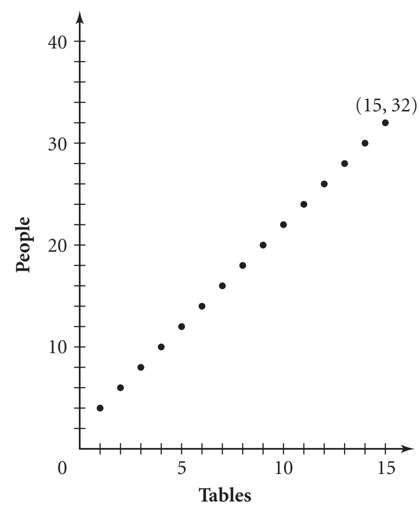
▸ ***Solution***



If you sketch the arrangements of four tables and five tables, you notice that when you add another table, you seat two more people than in the previous arrangement. You can put this information into a table, and that reveals a clear pattern. You can continue the pattern to find that 10 tables seat 22 people.

Tables	1	2	3	4	5	6	7	8	9	10
People	4	6	8	10	12	14	16	18	20	22

This graph shows the same information by plotting the points (1, 4), (2, 6), (3, 8), and so on. The graph also reveals a clear pattern. You can extend the graph to find that 15 tables are needed for 32 people.



You can use recursion—repeatedly adding 2 to the previous number of people—to find more numbers in the pattern or more points on the graph. A recursive definition tells the starting value and the mathematical operations that are required to find each subsequent value. For this example you could write

number of people at 1 table = 4

number of people at n tables = number of people at $(n - 1)$ tables + 2

This definition summarizes how to use recursion to find the number of people who can sit at any linear arrangement of tables. For example, to find how many people can sit at 10 tables, you take the number of people at 9 tables and add 2, or $20 + 2 = 22$.

A **sequence** is an ordered list of numbers. The table and graph in Example A represent the sequence

4, 6, 8, 10, 12, . . .

Each number in the sequence is called a **term.** The first term, u_1 (pronounced "u sub one"), is 4. The second term, u_2, is 6, and so on.

The nth term, u_n, is called the **general term** of the sequence. A **recursive formula,** the formula that defines a sequence, must specify one (or more) starting terms and a **recursive rule** that defines the nth term in relation to a previous term (or terms).

You generate the sequence 4, 6, 8, 10, 12, . . . with this recursive formula:

$$u_1 = 4$$
$$u_n = u_{n-1} + 2 \quad \text{where } n \geq 2$$

Because the starting value is $u_1 = 4$, the recursive rule $u_n = u_{n-1} + 2$ is first used to find u_2. This is clarified by saying that n must be greater than or equal to 2 to use the recursive rule.

This means *the first term is 4* and *each subsequent term is equal to the previous term plus 2.* Notice that each term, u_n, is defined in relation to the previous term, u_{n-1}. For example, the 10th term relies on the 9th term, or $u_{10} = u_9 + 2$.

EXAMPLE B

A concert hall has 59 seats in Row 1, 63 seats in Row 2, 67 seats in Row 3, and so on. The concert hall has 35 rows of seats. Write a recursive formula to find the number of seats in each row. How many seats are in Row 4? Which row has 95 seats?

An opera house in Sumter, South Carolina.

▶ Solution

First, it helps to organize the information in a table.

Row	1	2	3	4	. . .
Seats	59	63	67		. . .

Every recursive formula requires a starting term. Here the starting term is 59, the number of seats in Row 1. That is, $u_1 = 59$.

This sequence also appears to have a common difference between successive terms: 63 is 4 more than 59, and 67 is 4 more than 63. Use this information to write the recursive rule for the nth term, $u_n = u_{n-1} + 4$.

Therefore, this recursive formula generates the sequence representing the number of seats in each row:

$$u_1 = 59$$
$$u_n = u_{n-1} + 4 \quad \text{where } n \geq 2$$

You can use this recursive formula to calculate how many seats are in each row. [▶ See **Calculator Note 1B** to learn how to do recursion on your calculator. ◀]

$$u_1 = \boxed{59}$$ The starting term is 59.

$$u_2 = u_1 + 4 = \boxed{59} + 4 = \boxed{63}$$ Substitute 59 for u_1.

$$u_3 = u_2 + 4 = \boxed{63} + 4 = \boxed{67}$$ Substitute 63 for u_2.

$$u_4 = u_3 + 4 = \boxed{67} + 4 = 71$$ Continue using recursion.

$$\vdots$$

Because $u_4 = 71$, there are 71 seats in Row 4. If you continue the recursion process, you will find that $u_{10} = 95$, or that Row 10 has 95 seats.

In Example B, the terms of the sequence are related by a **common difference.** This type of sequence is called an **arithmetic sequence.**

Arithmetic Sequence

An **arithmetic sequence** is a sequence in which each term is equal to the previous term plus a constant. This constant is called the **common difference.** If d is the common difference, the recursive rule for the sequence has the form

$$u_n = u_{n-1} + d$$

The key to identifying an arithmetic sequence is recognizing the common difference. If you are given a few terms and need to write a recursive formula, first try subtracting consecutive terms. If $u_n - u_{n-1}$ is constant for each pair of terms, then you know your recursive rule must be a rule for an arithmetic sequence.

Investigation

Monitoring Inventory

Heater King, Inc., has purchased the parts to make 2000 water heaters. Each day, the workers assemble 50 water heaters from the available parts. The company has agreed to supply MegaDepot with 40 water heaters per day and Smalle Shoppe with 10 water heaters per day. MegaDepot currently has 470 water heaters in stock, and Smalle Shoppe has none. The management team at Heater King, Inc., needs a way to monitor inventory and demand.

Step 1 As a group, model what happens to the number of unmade water heaters, the inventory at MegaDepot, and the inventory at Smalle Shoppe. Keep track of your daily results in a table like this one. [▶ See **Calculator Note 1B** for different ways to do recursion on your calculator. ◀]

Day	Unmade water heaters	MegaDepot	Smalle Shoppe
1	2000	470	0
2			

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Step 2 Use your table from Step 1 to answer these questions:

a. How many days will it be until MegaDepot has an equal number or a greater number of water heaters than the number of water heaters left unmade?

b. The assembly-line machinery malfunctions the first day that MegaDepot has a number of water heaters greater than twice the number of unmade water heaters. How many water heaters does Smalle Shoppe have on the day that assembly stops?

Step 3 Write a short summary of how you modeled inventory and how you found the answers to the questions in Step 2. Compare your methods to the methods of other groups.

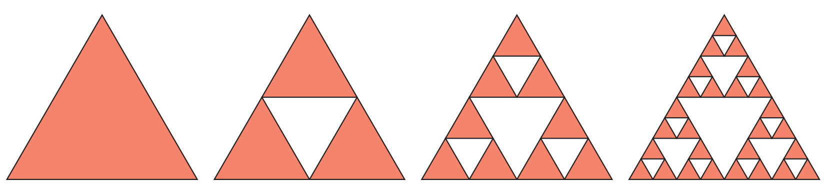
Career CONNECTION

Economics is the study of how goods and services are produced, distributed, and consumed. Economists in corporations, universities, and government agencies are concerned with the best way to meet human needs with limited resources. Professional economists use mathematics to study and model factors such as supply of resources, manufacturing costs, and selling price.

The sequences in Example A, Example B, and the investigation are arithmetic sequences. Example C introduces a different kind of sequence that is still defined recursively.

EXAMPLE C

The geometric pattern below is created recursively. If you continue the pattern endlessly, you create a **fractal** called the Sierpiński triangle. How many red triangles are there at Stage 20?



Stage 0 Stage 1 Stage 2 Stage 3

This stamp, part of Poland's 1982 "Mathematicians" series, portrays Waclaw Sierpiński.

Mathematics CONNECTION

The Sierpiński triangle is named after the Polish mathematician Waclaw Sierpiński (1882–1969). He was most interested in number theory, set theory, and topology, three branches of mathematics that study the relations and properties of sets of numbers or points. Sierpiński was highly involved in the development of mathematics in Poland between World War I and World War II. He published 724 papers and 50 books in his lifetime. He introduced his famous triangle pattern in a 1915 paper.

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▸ Solution

Count the number of red triangles at each stage and write a sequence.

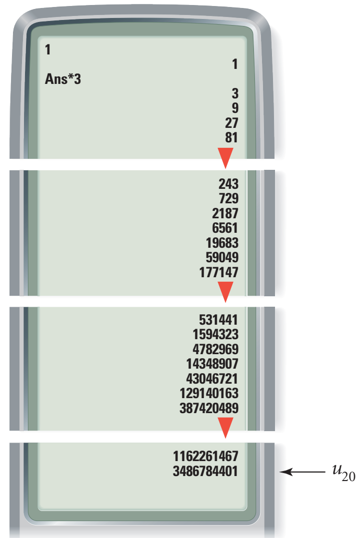
1, 3, 9, 27, . . .

The starting term, 1, represents the number of triangles at Stage 0. You can define the starting term as a zero term, or u_0. In this case, $u_0 = 1$.

Starting with the second term, each term of the sequence multiplies the previous term by 3, so that 3 is 3 times 1, 9 is 3 times 3, and 27 is 3 times 9. Use this information to write the recursive rule and complete your recursive formula.

$$u_0 = 1$$
$$u_n = 3 \cdot u_{n-1} \quad \text{where } n \geq 1$$

Using the recursive rule 20 times, you find that $u_{20} = 3{,}486{,}784{,}401$. There are over 3 billion triangles at Stage 20!

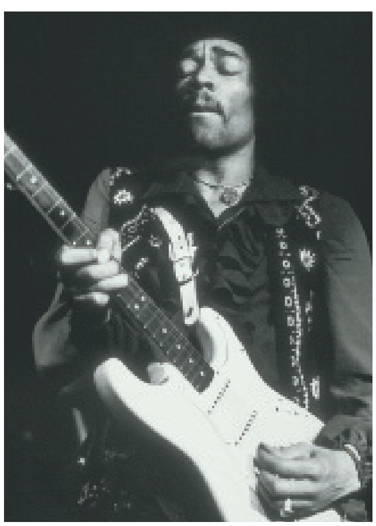


In Example C, consecutive terms of the sequence are related by a **common ratio.** This type of sequence is called a **geometric sequence.**

Geometric Sequence

A **geometric sequence** is a sequence in which each term is equal to the previous term multiplied by a constant. This constant is called the **common ratio.** If r is the common ratio, the recursive rule for the sequence has the form

$$u_n = r \cdot u_{n-1}$$



You identify a geometric sequence by dividing consecutive terms. If $\frac{u_n}{u_{n-1}}$ is constant for each pair of terms, then you know the sequence is geometric.

Arithmetic and geometric sequences are the most basic sequences because their recursive rules use only one operation: addition in the case of arithmetic sequences, and multiplication in the case of geometric sequences. Recognizing these basic operations will help you easily identify sequences and write recursive formulas.

Guitar feedback is a real-world example of recursion. When the amplifier is turned up loud enough, the sound is picked up by the guitar and amplified again and again, creating a feedback loop. Jimi Hendrix (1942–1970), a pioneer in the use of feedback and distortion in rock music, remains one of the most legendary guitar players of the 1960s.

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EXERCISES

You will need

Geometry software for Exercise 13

Practice Your Skills

1. Write the first 4 terms of each sequence.

a. $u_1 = 20$
$u_n = u_{n-1} + 6 \quad \text{where } n \geq 2$

b. $u_1 = 47$
$u_n = u_{n-1} - 3 \quad \text{where } n \geq 2$

c. $u_0 = 32$
$u_n = 1.5 \cdot u_{n-1} \quad \text{where } n \geq 1$

d. $u_1 = -18$
$u_n = u_{n-1} + 4.3 \quad \text{where } n \geq 2$

2. Identify each sequence in Exercise 1 as arithmetic or geometric. State the common difference or the common ratio for each.

3. Write a recursive formula and use it to find the missing table values.

n	1	2	3	4	5	. . .	
u_n	40	36.55	33.1	29.65		. . .	12.4

4. Write a recursive formula to generate an arithmetic sequence with a first term 6 and a common difference 3.2. Find the 10th term.

5. Write a recursive formula to generate each sequence. Then find the indicated term.

a. 2, 6, 10, 14, . . . Find the 15th term.
b. 10, 5, 0, −5, . . . Find the 12th term.
c. 0.4, 0.04, 0.004, 0.0004, . . . Find the 10th term.
d. −2, −8, −14, −20, −26, . . . Find the 30th term.
e. 1.56, 4.85, 8.14, 11.43, . . . Find the 14th term.
f. −6.24, −4.03, −1.82, 0.39, . . . Find the 20th term.

History CONNECTION

Hungarian mathematician Rózsa Péter (1905–1977) was the first person to propose the study of recursion in its own right. In an interview she described recursion in this way:

> The Latin technical term "recursion" refers to a certain kind of *stepping backwards* in the sequence of natural numbers, which necessarily ends after a finite number of steps. With the use of such recursions the values of even the most complicated functions used in number theory can be calculated in a finite number of steps.

In her book *Recursive Functions in Computer Theory,* Péter describes the important connections between recursion and computer languages.

Rózsa Péter

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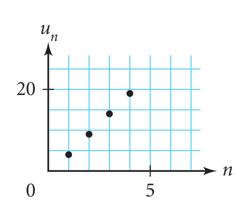
6. Write a recursive formula for the sequence graphed at right. Find the 46th term.



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Reason and Apply

7. Write a recursive formula that you can use to find the number of segments, u_n, for Figure n of this geometric pattern. Use your formula to complete the table.

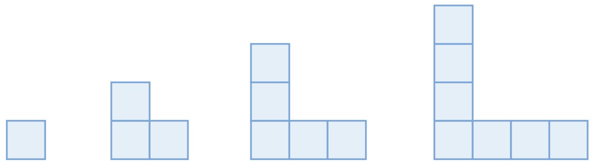


Figure 1 Figure 2 Figure 3 Figure 4

Figure	1	2	3	4	5	. . .	12	. . .	
Segments	4	10	16			. . .		. . .	190

8. A 50-gallon (gal) bathtub contains 20 gal of water and is filling at a rate of 2.4 gal/min. You check the tub every minute on the minute.

a. Suppose that the drain is closed. When will you discover that the water is flowing over the top?

b. Now suppose that the bathtub contains 20 gal of water and is filling at a rate of 2.4 gal/min, but the drain is open and water drains at a rate of 3.1 gal/min. When will you discover that the tub is empty?

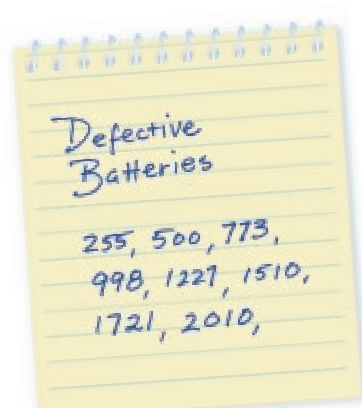
c. Write a recursive formula that you can use to find the water level at any minute due to both the rate of filling and the rate of draining.

9. A car leaves town heading west at 57 km/h.

a. How far will the car travel in 7 h?

b. A second car leaves town 2 h after the first car, but it is traveling at 72 km/h. To the nearest hour, when will the second car pass the first?

10. **APPLICATION** Inspector 47 at the Zap battery plant keeps a record of which AA batteries she finds defective. Although the battery numbers at right do not make an exact sequence, she estimates an arithmetic sequence.



a. Write a recursive formula for an arithmetic sequence that estimates which batteries are defective. Explain your reasoning.

b. Predict the numbers of the next five defective batteries.

c. How many batteries in 100,000 will be defective?

11. The week of February 14, the owner of Nickel's Appliances stocks hundreds of red, heart-shaped vacuum cleaners. The next week, he still has hundreds of red, heart-shaped vacuum cleaners. He tells the manager, "Discount the price 25 percent each week until they are gone."

a. On February 14, the vacuums are priced at $80. What is the price of a vacuum during the second week?

b. What is the price during the fourth week?

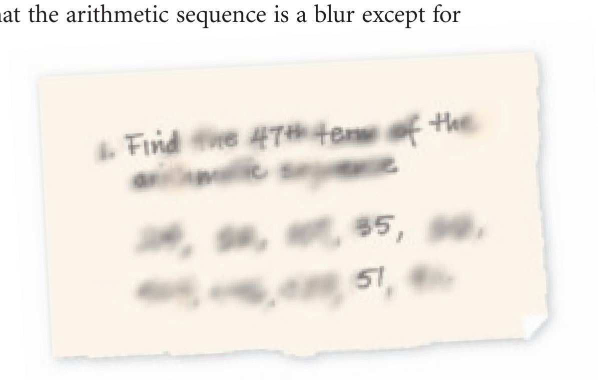
c. When will the vacuum sell for less than $10?

12. Taoufik picks up his homework paper from the puddle it has fallen in. Sadly he reads the first problem and finds that the arithmetic sequence is a blur except for two terms.

a. What is the common difference? How did you find it?

b. What are the missing terms?

c. What is the answer to Taoufik's homework problem?



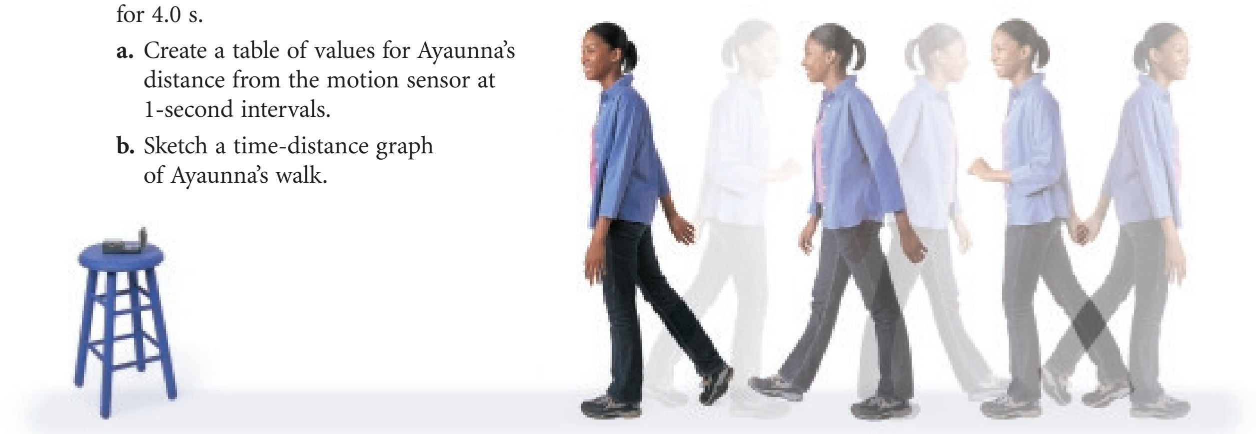
13. *Technology* Use geometry software to construct 10 segments whose lengths represent the first 10 terms of a sequence. Describe how you constructed the segments, and explain whether your sequence is arithmetic or geometric.

Review

14. Ayaunna starts 2.0 m from a motion sensor. She walks away from the sensor at a rate of 1.0 m/s for 3.0 s and then walks toward the sensor at a rate of 0.5 m/s for 4.0 s.

a. Create a table of values for Ayaunna's distance from the motion sensor at 1-second intervals.

b. Sketch a time-distance graph of Ayaunna's walk.



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15. Write each question as a proportion and then find the unknown number.

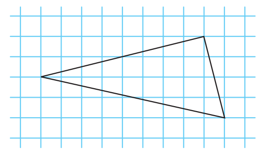
a. 70% of 65 is what number?

b. 115% of 37 is what number?

c. 110 is what percent of 90?

d. What percent of 18 is 0.5?

16. Find the area of this triangle using two different strategies. Describe your strategies.



17. **APPLICATION** Sherez is currently earning \$390 per week as a store clerk and part-time manager. She is offered either a 7% increase or an additional \$25 per week. Which offer should she accept?

IMPROVING YOUR REASONING SKILLS

Fibonacci and the Rabbits

Suppose a newborn pair of rabbits, one male and one female, is put in a field. Assume that rabbits are able to mate at the age of one month, so at the end of its second month a female can produce another pair of rabbits. Suppose that each female who is old enough produces one new pair of rabbits (one male, one female) every month and that none of the rabbits die. Write the first few terms of a sequence that shows how many pairs there will be at the end of each month. Then write a recursive formula for the sequence.

This sequence is called the **Fibonacci sequence** after Italian mathematician Leonardo Fibonacci (ca. 1170–1240), who asked a similar problem in his book *Liber abaci* (1202). How is the Fibonacci sequence unique compared to the other sequences you have studied?

Two arctic hares blend with the white tundra in Ellesmere Island National Park in northern Canada.

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LESSON 1.2

Modeling Growth and Decay

Each sequence you generated in Lesson 1.1 was either an arithmetic sequence with a recursive rule in the form $u_n = u_{n-1} + d$ or a geometric sequence with a recursive rule in the form $u_n = r \cdot u_{n-1}$. You compared consecutive terms to decide whether the sequence required a common difference or a common ratio.

In most cases you have used u_1 as the starting term of each sequence. In some situations (like the one in the next investigation), it is more meaningful to treat the starting term as a zero term, or u_0. The zero term represents the starting value before any change occurs. You can decide whether it would be better to begin at u_0 or u_1.

EXAMPLE A

An automobile depreciates, or loses value, as it gets older. Suppose that a particular automobile loses one-fifth of its value each year. Write a recursive formula to find the value of this car when it is 6 years old, if it cost $23,999 when it was new.

Consumer CONNECTION

The *Kelley Blue Book,* first compiled in 1926 by Les Kelley, annually publishes standard values of every vehicle on the market. Many people who want to know the value of an automobile will ask what its "Blue Book" value is. The *Kelley Blue Book* calculates the value of a car by accounting for its make, model, year, mileage, location, and condition.

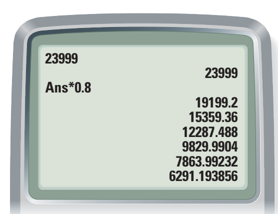
▶ *Solution*

Each year, the car will be worth $\frac{4}{5}$ of what it was worth the previous year, so the recursive sequence is geometric. It is convenient to start with $u_0 = 23999$ to represent the value of the car when it was new so that u_1 will represent the value after one year, and so on. The recursive formula that generates the sequence of annual values is

$$u_0 = 23999 \qquad \text{Starting value.}$$
$$u_n = 0.8 \cdot u_{n-1} \quad \text{where } n \geq 1 \qquad \tfrac{4}{5} \text{ is } 0.8.$$

Use this rule to find the 6th term.

After 6 years, the car is worth $6,291.19.



In situations like the problem in Example A, it's easier to write a recursive formula than an equation using x and y.

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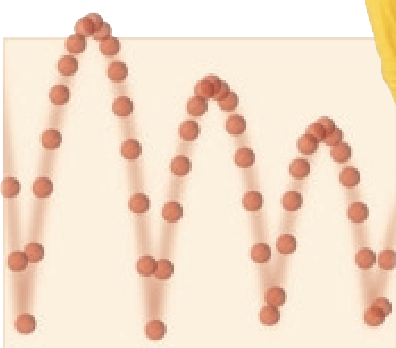
Investigation
Looking for the Rebound

You will need
- a ball
- a motion sensor

When you drop a ball, the rebound height becomes smaller after each bounce. In this investigation you will write a recursive formula for the height of a real ball as it bounces.



Procedure Note

Collecting Data
1. Hold the motion sensor above the ball.
2. Press the trigger, then release the ball.
3. If the ball drifts, try to follow it and maintain the same height with the motion sensor.
4. If you do not capture at least 6 good consecutive bounces, repeat the procedure.

Step 1 Set up your calculator and motion sensor and follow the Procedure Note to collect bouncing-ball data. [▶ See **Calculator Note 1F** for calculator instructions on how to gather data. ◀]

Step 2 The data transferred to your calculator are in the form (x, y), where x is the time since you pressed the trigger, and y is the height of the ball. Trace the data graphed by your calculator to find the starting height and the rebound height after each bounce. Record your data in a table.

Step 3 Graph a scatter plot of points in the form (*bounce number, rebound height*). [▶ See **Calculator Notes 1G, 1H, 1I,** and **1J** to learn how to enter, graph, trace, and share data. ◀]

Step 4 Compute the rebound ratio for consecutive bounces.

$$\textit{rebound ratio} = \frac{\textit{rebound height}}{\textit{previous rebound height}}$$

Step 5 Decide on a single value that best represents the rebound ratio for your ball. Use this ratio to write a recursive formula that models your sequence of *rebound height* data, and use it to generate the first six terms.

Step 6 Compare your experimental data to the terms generated by your recursive formula. How close are they? Describe some of the factors that might affect this experiment. For example, how might the formula change if you use a different kind of ball?

You may find it easier to think of the common ratio as the whole, 1, plus or minus a percent change. In place of r you can write $(1 + p)$ or $(1 - p)$. The car example involved a 20% (one-fifth) loss, so the common ratio could be written as $(1 - 0.20)$. Your bouncing ball may have had a common ratio of 0.75, which you can write as $(1 - 0.25)$ or a 25% loss per bounce. These are examples of decay, or geometric sequences that decrease. The next example is one of growth, or a geometric sequence that increases.

EXAMPLE B

Gloria deposits \$2000 into a bank account that pays 7% annual interest compounded annually. This means the bank pays her 7% of her account balance as interest at the end of each year, and she leaves the original amount and the interest in the account. When will the original deposit double in value?

Economics CONNECTION

Interest is a charge that you pay for borrowing money, or that the bank pays you for letting them invest the money you keep in your bank account. Simple interest is a percentage paid on the **principal,** or initial balance, over a period of time. If you leave the interest in the account, then in the next time period you receive interest on both the principal and the interest that were in your account. This is called **compound interest** because you are receiving interest on the interest.

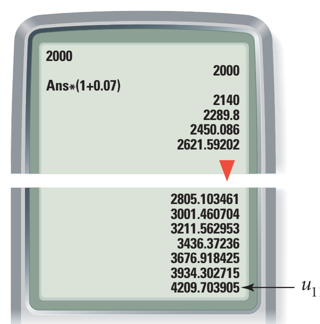
▸ Solution

The balance starts at \$2000 and increases by 7% each year.

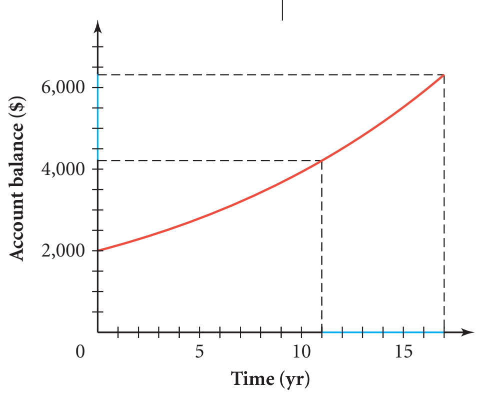
$$u_0 = 2000$$
$$u_n = (1 + 0.07)u_{n-1} \quad \text{where } n \geq 1$$

The recursive rule that represents 7% growth.

Use your calculator to compute year-end balances recursively.



The 11th term, u_{11}, is 4209.70, so the investment balance will more than double in 11 years.



Compound interest has many applications in everyday life. The interest on both savings and loans is almost always compounded, often leading to surprising results. This graph shows the account balance in the previous example.

Leaving just \$2000 in the bank at a good interest rate for 11 years can double your money. In another 6 years, the money will triple.

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Some banks will compound the interest monthly. You can write the common ratio as $\left(1 + \frac{0.07}{12}\right)$ to represent one-twelfth of the interest, compounding monthly. When you do this, n represents months instead of years. How would you change the rule to show that the interest is compounded 52 times per year? What would n represent in this situation?

EXERCISES

Practice Your Skills

1. Find the common ratio for each sequence.

a. 100, 150, 225, 337.5, 506.25 . . .

b. 73.4375, 29.375, 11.75, 4.7, 1.88 . . .

c. 80.00, 82.40, 84.87, 87.42, 90.04 . . .

d. 208.00, 191.36, 176.05, 161.97 . . .

2. Identify each sequence in Exercise 1 as growth or decay. Give the percent change for each.

3. Write a recursive formula for each sequence in Exercise 1 and find the 10th term. Use u_1 for the first term given.

4. Match each recursive rule to a graph. Explain your reasoning.

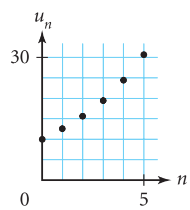
A. $u_1 = 10$
$u_n = (1 - 0.25) \cdot u_{n-1}$ where $n \geq 2$

B. $u_1 = 10$
$u_n = (1 + 0.25) \cdot u_{n-1}$ where $n \geq 2$

C. $u_1 = 10$
$u_n = 1 \cdot u_{n-1}$ where $n \geq 2$

Films quickly display a sequence of photographs, creating an illusion of motion.

i.



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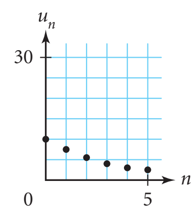

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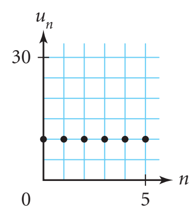


Keymath.com More Practice Your Skills

5. Factor these expressions so that the variable appears only once.

a. $u_{n-1} + 0.07u_{n-1}$ **b.** $A - 0.18A$ **c.** $x + 0.08125x$ **d.** $2u_{n-1} - 0.85u_{n-1}$

Reason and Apply

6. Suppose the initial height from which a rubber ball drops is 100 in. The rebound heights to the nearest inch are 80, 64, 51, 41, . . .

a. What is the rebound ratio for this ball?

b. What is the height of the tenth rebound?

c. After how many bounces will the ball rebound less than 1 in.? Less than 0.1 in.?

7. Suppose the recursive forumula $u_0 = 100$ and $u_n = (1 - 0.20)u_{n-1}$ where $n \geq 1$ models a bouncing ball. Give real-world meanings for the numbers 100 and 0.20.

8. Suppose the recursive formula $u_{2003} = 250000$ and $u_n = (1 + 0.025)u_{n-1}$ where $n \geq 2004$ describes an investment made in the year 2003. Give real-world meanings for the numbers 250,000 and 0.025.

9. **APPLICATION** A small company with 12 employees is growing at a rate of 20% per year. It will need to hire more employees to keep up with the growth, assuming its business keeps growing at the same rate.

a. How many people should the company plan to hire in each of the next five years?

b. How many employees will it have in five years?

Economics CONNECTION

To manage the financial demands of a company's growth, forecasters use the substantial growth rate model. This rate, represented by the variable g^*, is the annual percentage increase in sales that a company can maintain while keeping its capital stable. If a company grows at a rate faster than g^*, it may not have the means to finance its growth. If it grows slowly, its capital will grow at a pace that will reduce debt and increase investments. Using g^* allows a business to project its financial ability to grow.

10. **APPLICATION** The table below shows investment balances over time.

Elapsed time (yr)	0	1	2	3	. . .
Balance ($)	2000	2170	2354.45	2554.58	. . .

a. Write a recursive formula that generates the balances in the table.

b. What is the annual interest rate?

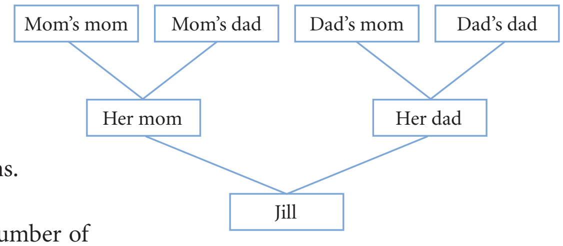
c. How many years will it take before the original deposit triples in value?

11. **APPLICATION** Carbon dating is used to find the age of ancient remains of once-living things. Carbon-14 is found naturally in all living things, and it decays slowly after death. About 11.45% of it decays in each 1000-year period of time.

Let 100%, or 1, be the beginning amount of carbon-14. At what point will less than 5% remain? Write the recursive formula you used.

At an excavation site in Alberta, Canada, these scientists uncover the remains of an Albertosaurus (a relative of the Tyrannosaurus) about 65–70 million years old.

12. Suppose Jill's biological family tree looks like the diagram at right. You can model recursively the number of people in each generation.



a. Make a table showing the number of Jill's ancestors in each of the past five generations. Use u_0 to represent Jill's generation.

b. Look in your table at the sequence of the number of ancestors. Describe how to find u_n if you know u_{n-1}. Write a recursive formula.

c. Find the number of the term of this sequence that is closest to 1 billion. What is the real-world meaning of this answer?

d. If a new generation is born every 25 years, approximately when did Jill have 1 billion living ancestors in the same generation?

e. Your answer to 12c assumes there are no duplicates, that is, no common ancestors on Jill's mom's and Jill's dad's sides of the family. Look up Earth's population for the year you found in 12d. You will find helpful links at www.keymath.com/DAA. Write a few sentences describing any problems you notice with the assumption of no common ancestors.

Cultural CONNECTION

Family trees are lists of family descendants and are used in the practice of genealogy. People who research genealogy may want to trace their family's medical history or national origin, discover important dates, or simply enjoy it as a hobby. Alex Haley's 1976 genealogical book, *Roots: The Saga of an American Family,* told the powerful history of his family's prolonged slavery and decades of discrimination. The book, along with the 1977 television miniseries, inspired many people to trace their family lineage.

Alex Haley (1921–1992)

13. **APPLICATION** Suppose $500 is deposited into an account that earns 6.5% annual interest and no more deposits or withdrawals are made.

a. If the interest is compounded monthly, what is the monthly rate?

b. What is the balance after 1 month?

c. What is the balance after 1 year?

d. What is the balance after 29 months?

14. **APPLICATION** Between 1970 and 2000, the population of Grand Traverse County in Michigan grew from 39,175 to 77,654.

a. Find the percent increase over the 30-year period.

b. What do you think the *annual* growth rate was during this period?

c. Check your answer to 14b by using a recursive formula. Do you get 77,654 people after 30 years? Explain why your recursive formula may not work.

d. Use guess-and-check to find a growth rate, to the nearest 0.1% (or 0.001), that comes closest to producing the 30-year growth experienced.

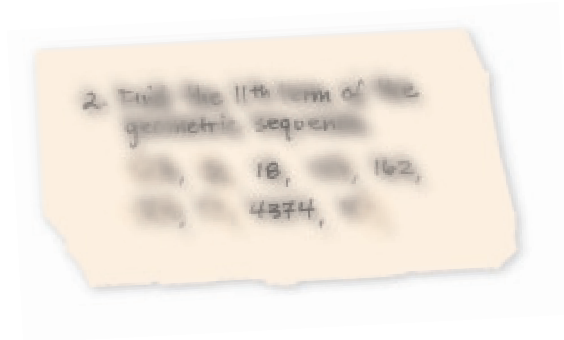
e. Use your answer to 14d to estimate the population in 1985. How does this compare with the average of the populations of 1970 and 2000? Why is that?

15. Taoufik looks at the second problem of his wet homework that had fallen in a puddle.

a. What is the common ratio? How did you find it?

b. What are the missing terms?

c. What is the answer he needs to find?



Review

16. The population of the United States grew 13.20% from 1990 to 2000. The population reported in the 2000 census was 281.4 million. What population was reported in 1990? Explain how you found this number.

17. An elevator travels at a nearly constant speed from the ground to an observation deck at 160 m. This trip takes 40 s. The trip back down is also at this same constant speed.

a. What is the elevator's speed in meters per second?

b. How long does it take the elevator to reach the restaurants, located 40 m above ground level?

c. Graph the height of the elevator as it moves from ground level to the observation deck.

d. Graph the height of the elevator as it moves from the restaurant level, at 40 m, to the observation deck.

e. Graph the height of the elevator as it moves from the deck to ground level.

The CN Tower in Toronto is one of Canada's landmark structures and one of the world's tallest buildings. Built in 1976, it has six glass-fronted elevators that allow you to view the landscape as you rise above it at 15 mi/h. At 1136 ft, you can either brace against the wind on the outdoor observation deck or test your nerves by walking across a 256 ft^2 glass floor with a view straight down.

18. Consider the sequence 180, 173, 166, 159, ...

a. Write a recursive formula. Use $u_1 = 180$.

b. What is u_{10}?

c. What is the first term with a negative value?

19. Solve each equation.

a. $-151.7 + 3.5x = 0$ **b.** $0.88x + 599.72 = 0$

c. $18.75x - 16 = 0$ **d.** $0.5 \cdot 16 + x = 16$

20. For the equation $y = 47 + 8x$, find the value of y when

a. $x = 0$ **b.** $x = 1$

c. $x = 5$ **d.** $x = -8$

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LESSON

1.3

Keymath.com Links to Resources

A First Look at Limits

A Mathematician is a machine for turning coffee into theorems.

PAUL ERDÖS

Increasing arithmetic and geometric sequences, such as the number of new triangles at each stage in a Sierpiński triangle or the balance of money earning interest in the bank, have terms that get larger and larger. But can a tree continue to grow larger year after year? Can people continue to build taller buildings, run faster, and jump higher, or is there a limit to any of these?

The women's world record for the fastest time in the 100 m dash has decreased by about 3 s in 66 yr. Marie Mejzlíková (Czechoslovakia) set the record at 13.6 s in 1922, and Florence Griffith-Joyner (USA), shown at right, set it at 10.49 s in 1988. In the 1998 article "How Good Can We Get?" Jonas Mureika predicts that the ultimate performance for a woman in the 100 m dash will be 10.15 s.

How large can a tree grow? It depends partly on environmental factors such as disease and climate. Trees have mechanisms that slow their growth as they age, similar to human growth. (Unlike humans, however, a tree may not reach maturity until 100 yr after it starts growing.) This giant sequoia, the General Sherman Tree in Sequoia National Park, California, is considered to be the world's largest living thing. The volume of its trunk is over 52,500 ft^3.

Decreasing sequences may also have a limit. For example, the temperature of a cup of hot cocoa as it cools, taken at one-minute intervals, produces a sequence that approaches the temperature of the room. In the long run, the hot cocoa will be at room temperature.

In the next investigation you will explore what happens to a sequence in the long run.

Investigation
Doses of Medicine

You will need

- a bowl
- a supply of water
- a supply of tinted liquid
- measuring cups, graduated in milliliters
- a sink or waste bucket

Our kidneys continuously filter our blood, removing impurities. Doctors take this into account when prescribing the dosage and frequency of medicine.

In this investigation you will simulate what happens in the body when a patient takes medicine. To represent the blood in a patient's body, use a bowl containing a total of 1 liter (L) of liquid. Start with 16 milliliters (mL) of tinted liquid to represent a dose of medicine in the blood, and use clear water for the rest.

Step 1 Suppose a patient's kidneys filter out 25% of this medicine each day. To simulate this, remove $\frac{1}{4}$, or 250 mL of the mixture from the bowl and replace it with 250 mL of clear water to represent filtered blood. Make a table like the one below, and record the amount of medicine in the blood over several days. Repeat the simulation for each day.

Step 2 Write a recursive formula that generates the sequence in your table.

Step 3 How many days will pass before there is less than 1 mL of medicine in the blood?

Step 4 Is the medicine ever completely removed from the blood? Why or why not?

Step 5 Sketch a graph and describe what happens in the long run.

Day	Amount of medicine (mL)
0	16
1	
2	
3	

A single dose of medicine is often not enough. Doctors prescribe regular doses to produce and maintain a high enough level of medicine in the body. Next you will modify your simulation to look at what happens when a patient takes medicine daily over a period of time.

Step 6 Start over with 1 L of liquid. Again, all of the liquid is clear water, representing the blood, except for 16 mL of tinted liquid to represent the initial dose of medicine. Each day, 250 mL of liquid is removed and replaced with 234 mL of clear water and 16 mL of tinted liquid to represent a new dose of medicine. Complete another table like the one in Step 1, recording the amount of medicine in the blood over several days.

Step 7 Write a recursive formula that generates this sequence.

Step 8 Do the contents of the bowl ever turn into pure medicine? Why or why not?

Step 9 Sketch a graph and explain what happens to the level of medicine in the blood after many days.

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Medicine and its elimination from the human body is a real-world example of a dynamic, or changing, system. A contaminated lake and its cleanup processes is another real-world example. Dynamic systems often reach a point of stability in the long run. The quantity associated with that stability, such as the number of milliliters of medicine, is called a **limit.** Mathematically, we say that the sequence of numbers associated with the system approaches that limit. Being able to predict limits is very important for analyzing these situations. The long-run value helps you estimate limits.

Environmental CONNECTION

The Cuyahoga River in Cleveland, Ohio, caught fire several times during the 1950s and 1960s because its water was so polluted with volatile chemicals. The events inspired several clean water acts in the 1970s and the creation of the federal Environmental Protection Agency. After testing the toxic chemicals present in the water and locating possible sources of contamination, environmental engineers established pollution control levels and set standards for monitoring waste from local industries. In the end, the private and corporate sectors of Cleveland managed to clean up the waterways, preserve the wildlife areas that relied on them, and provide a number of parks for recreational use.

This photograph shows a fire on the Cuyahoga River in 1952.

Each of the sequences in the investigation approached different long-run values. The first sequence approached zero. The second sequence was shifted and it approached a nonzero value. A **shifted geometric sequence** includes an added term in the recursive rule. Let's look at another example of a shifted geometric sequence.

EXAMPLE

Antonio and Deanna are working at the community pool for the summer. They need to provide a "shock" treatment of 450 grams (g) of dry chlorine to prevent the growth of algae in the pool, then they add 45 g of chlorine each day after the initial treatment. Each day, the sun burns off 15% of the chlorine. Find the amount of chlorine after 1 day, 2 days, and 3 days. Create a graph that shows the chlorine level after several days and in the long run.

▶ ***Solution***

The starting value is given as 450. This amount decays by 15% a day, but 45 g is also added each day. The amount remaining after each day is generated by the rule $u_n = (1 - 0.15)u_{n-1} + 45$, or $u_n = 0.85u_{n-1} + 45$. Use this rule to find the chlorine level in the long run.

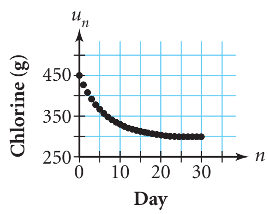
$u_0 = 450$ — The initial shock treatment.

$u_1 = 0.85(450) + 45 = 427.5$ — The amount after 1 day.

$u_2 = 0.85(427.5) + 45 \approx 408.4$ — The amount after 2 days.

$u_3 = 0.85(408.4) + 45 \approx 392.1$ — The amount after 3 days.

To find the long-run value of the amount of chlorine, you can continue evaluating terms until the value stops changing, or see where the graph levels off. From the graph, the long-run value appears to be 300 g of chlorine.



You can also use algebra to find the value of the terms as they level off. If you assume that terms stop changing, then you can set the value of the next term equal to the value of the previous term and solve the equation.

$u_n = 0.85u_{n-1} + 45$	Recursive rule.
$c = 0.85c + 45$	Assign the same variable to u_n and u_{n-1}.
$0.15c = 45$	Subtract $0.85c$ from both sides.
$c = 300$	Divide both sides by 0.15.

The amount of chlorine will level off at 300 g, which agrees with the long-run value estimated from the graph.

The study of limits is an important part of calculus, the mathematics of change. Understanding limits mathematically will give you a chance to work with other real-world applications in biology, chemistry, physics, and social science.

EXERCISES

Practice Your Skills

1. Find the value of u_1, u_2, and u_3. Identify the type of sequence (arithmetic, geometric, or shifted geometric) and tell whether it is increasing or decreasing.

a. $u_0 = 16$
$u_n = (1 - 0.05)u_{n-1} + 16 \quad$ where $n \geq 1$

b. $u_0 = 800$
$u_n = (1 - 0.05)u_{n-1} + 16 \quad$ where $n \geq 1$

c. $u_0 = 50$
$u_n = (1 - 0.10)u_{n-1} \quad$ where $n \geq 1$

d. $u_0 = 40$
$u_n = (1 - 0.50)u_{n-1} + 20 \quad$ where $n \geq 1$

2. Solve each equation.

a. $a = 210 + 0.75a$ **b.** $b = 0.75b + 300$ **c.** $c = 210 + c$ **d.** $d = 0.75d$

3. Find the long-run value for each sequence in Exercise 1.

Keymath.com More Practice Your Skills

4. Write a recursive formula for each sequence.

a. 200.00, 216.00, 233.28, 251.94 . . . **b.** 0, 10, 15, 17.5, 18.75 . . .

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Reason and Apply

5. The Osbornes have a small pool and are doing a chlorine treatment. The recursive formula below gives the pool's daily amount of chlorine in grams.

$u_0 = 300$

$u_n = (1 - 0.15)u_{n-1} + 30 \quad \text{where } n \geq 1$

a. Explain the real-world meanings of the values 300, 0.15, and 30 in this formula.

b. Describe what happens to the chlorine level in the long run.

6. **APPLICATION** On October 1, 2002, Sal invested \$24,000 in a bank account earning 3.4% annually, compounded monthly. A month later, he withdrew \$100 and continued to withdraw \$100 on the first of every month thereafter.

a. Write a recursive formula for this problem.

b. List the first five terms of this sequence of balances, starting with the initial investment.

c. What is the meaning of the value of u_4?

d. What is the balance on October 2, 2003? On October 2, 2005?

7. **APPLICATION** Consider the bank account in Exercise 6.

a. What happens to the balance if the same interest and withdrawal patterns continue for a long time? Does the balance ever level off?

b. What monthly withdrawal amount would maintain a constant balance of \$24,000 in the long run?

8. **APPLICATION** The Forever Green Nursery owns 7000 white pine trees. Each year, the nursery plans to sell 12% of its trees and plant 600 new ones.

a. Find the number of pine trees owned by the nursery after 10 years.

b. Find the number of pine trees owned by the nursery after many years, and explain what is occurring.

c. What equation can you solve to find the number of trees in the long run?

d. Try different starting totals in place of the 7000 trees. Describe any changes to the long-run value.

e. In the fifth year, a disease destroys many of the nursery's trees. How does the long-run value change?

9. **APPLICATION** Jack takes a capsule containing 20 milligrams (mg) of a prescribed allergy medicine early in the morning. By the same time a day later, 25% of the medicine has been eliminated from his body. Jack doesn't take any more medicine, and his body continues to eliminate 25% of the remaining medicine each day. Write a recursive formula for the daily amount of this medicine in Jack's body. When will there be less than 1 mg of the medicine remaining in his body?

10. Consider the last part of the Investigation Doses of Medicine. If you double the amount of medicine taken each time from 16 mL to 32 mL, but continue to filter only 250 mL of liquid, will the limit of the concentration be doubled? Explain.

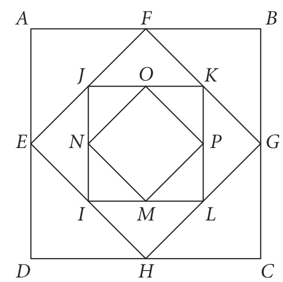
11. **APPLICATION** An anti-asthmatic drug has a half-life of about 9 hours. This means that 9 hours is the length of time it takes for the amount of drug present in a person's blood to decrease to half that amount.

a. Explain what this means about the amount of this drug in a person's bloodstream that starts out with a drug concentration of 16 mg/L.

b. Create a graph of points in the form (*elapsed time, drug concentration*) using 9-hour increments on the time axis.

c. What dosage of this drug should a person take every 9 hours to maintain a balance of at least 16 mg/L?

12. Suppose square *ABCD* with side length 8 in. is cut from paper. Another square, *EFGH,* is placed with its corners at the midpoints of *ABCD,* as shown. A third square is placed with its corners at midpoints of *EFGH,* and so on.



a. What is the perimeter of the ninth square?

b. What is the area of the ninth square?

c. What happens to the ratio of perimeter to area as the squares get smaller?

Review

13. Assume two terms of a sequence are $u_3 = 16$ and $u_4 = 128$.

a. Find u_2 and u_5 if the sequence is arithmetic. **b.** Find u_2 and u_5 if the sequence is geometric.

14. A park biologist estimates the moose population in a national park over a four-year period of mild winters. She makes this table.

Year	Estimated number of moose
1998	760
1999	835
2000	920
2001	1010

a. Write a recursive formula that approximately models the growth in the moose population for this four-year period.

b. The winter of 2002 was particularly severe, and the park biologist has predicted a decline of 10% to 15% in the moose herd. What is the range of moose population she predicts for 2002?

15. If a rubber ball rebounds to 97% of its height with each bounce, how many times will it bounce before it rebounds to half its original height?

IMPROVING YOUR VISUAL THINKING SKILLS

Think Pink

You have two 1-gallon cans. One contains 1 gallon of white paint, and the other contains 3 quarts (qt) of red paint. (There are 4 quarts per gallon.) You pour 1 qt of white paint into the red, mix it, and then pour 1 qt of the mixture back into the can of white paint. What is the red-white content of each can now? If you continually repeat the process, when will the two cans be the same shade of pink?

LESSON 1.4

Keymath.com Links to Resources

Graphing Sequences

By looking for numerical patterns, you can write a recursive formula that generates a sequence of numbers quickly and efficiently. You can also use graphs to help you identify patterns in a sequence.

Investigation
Match Them Up

Below are 18 different representations of sequences. Match each table with a recursive formula and a graph that represent the same sequence. As you do the matching, think about similarities and differences between the sequences and how those similarities and differences affect the tables, formulas, and graphs.

1.

n	u_n
0	8
1	4
3	1
6	0.125
9	0.015625

2.

n	u_n
0	0.5
1	1
2	2
3	4
4	8

3.

n	u_n
0	−2
1	1
2	2.5
4	3.625
5	3.8125

4.

n	u_n
0	−2
2	2
5	8
7	12
10	18

5.

n	u_n
0	8
1	6
3	2
5	−2
7	−6

6.

n	u_n
0	−4
1	−4
2	−4
4	−4
8	−4

A. $u_0 = 8$
$u_n = u_{n-1} - 2 \quad \text{where } n \geq 1$

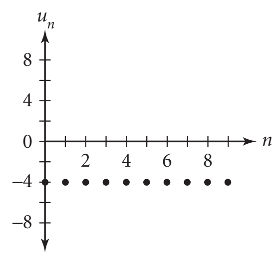
B. $u_0 = 8$
$u_n = 0.5u_{n-1} \quad \text{where } n \geq 1$

C. $u_0 = 0.5$
$u_n = 2u_{n-1} \quad \text{where } n \geq 1$

D. $u_0 = -2$
$u_n = u_{n-1} + 2 \quad \text{where } n \geq 1$

E. $u_0 = -4$
$u_n = u_{n-1} \quad \text{where } n \geq 1$

F. $u_0 = -2$
$u_n = 0.5u_{n-1} + 2 \quad \text{where } n \geq 1$

i.


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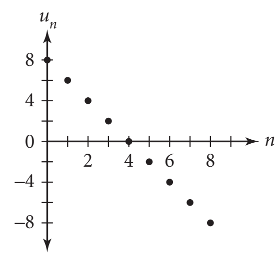

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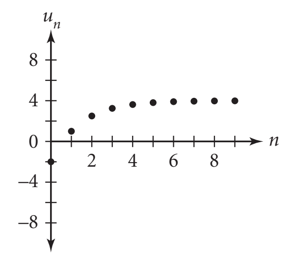

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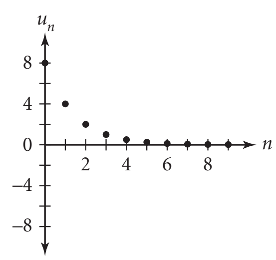

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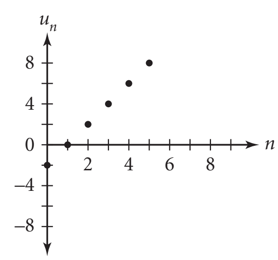

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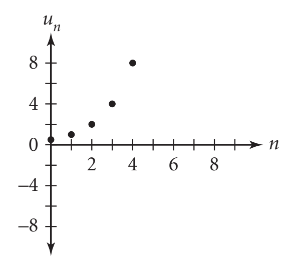

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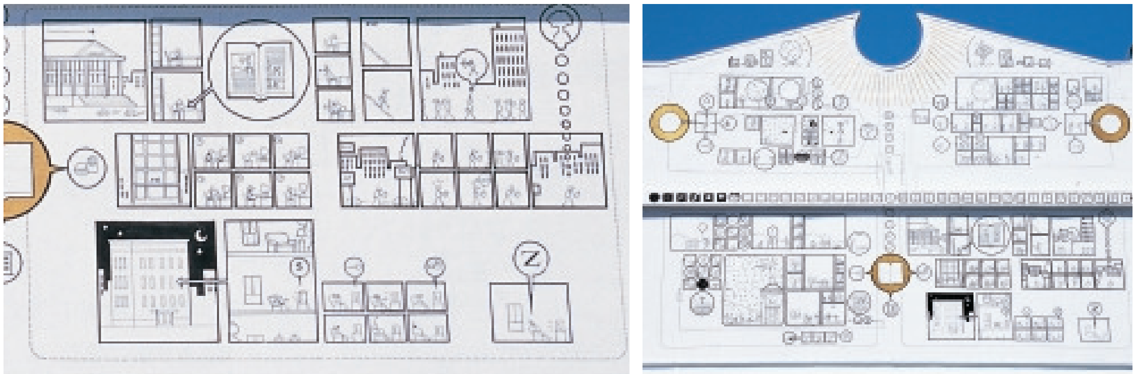
Write a paragraph that summarizes the relationships between different types of sequences, different types of recursive formulas, and different types of graphs. What generalizations can you make? What do you notice about the shapes of the graphs created from arithmetic and geometric sequences?

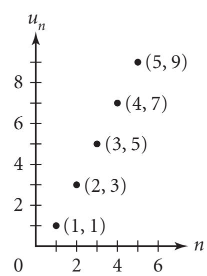


Many cartoons and comic strips show a sequence of events in linear order. The artwork of American artist Chris Ware (b 1967) breaks the convention by showing many sequences intertwined. This complex mural by Ware, at 826 Valencia Street in San Francisco, depicts human development of written and spoken communication.

The general shape of the graph of a sequence's terms gives you an indication of the type of sequence necessary to generate the terms.

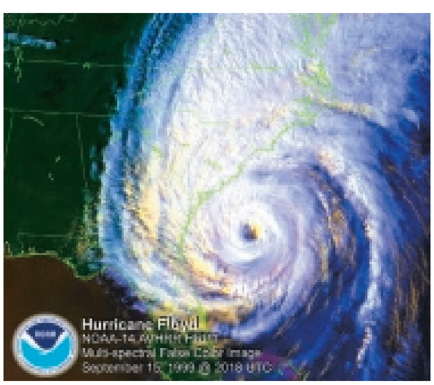
The graph at right is a visual representation of the first five terms of the arithmetic sequence generated by the recursive formula

$$u_1 = 1$$
$$u_n = u_{n-1} + 2 \quad \text{where } n \geq 2$$



This graph, in particular, appears to be **linear,** that is, the points appear to lie on a line. The common difference, $d = 2$, makes each new point rise 2 units above the previous point.

Graphs of sequences are examples of **discrete graphs,** or graphs made of isolated points. It is incorrect to connect those isolated points with a continuous line or curve because the term number, n, must be a whole number.



The general shape of the graph of a sequence allows you to recognize whether the sequence is arithmetic or geometric. Even if the graph represents data that are not generated by a sequence, you may be able to find a sequence that is a **model,** or a close fit, for the data. The more details you can identify from the graph, the better you will be at fitting a model.

Weather forecasting is one career that relies on mathematical modeling. Forecasters use computers and sophisticated models to monitor changes in the atmosphere. Trends in the data can help predict the trajectory and severity of an impending storm, such as a hurricane.

EXAMPLE

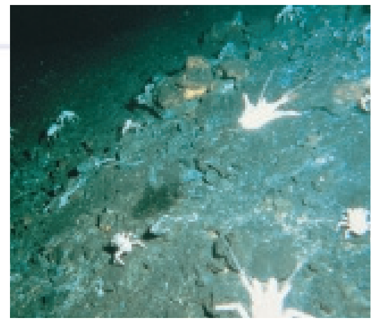
In deep water, divers find that their surroundings become darker the deeper they go. The data here give the percent of surface light intensity that remains at depth n ft in a particular body of water.

Depth (ft)	0	10	20	30	40	50	60	70
Percent of surface light	100	78	60	47	36	28	22	17

Find a sequence model that approximately fits these data.

Science CONNECTION

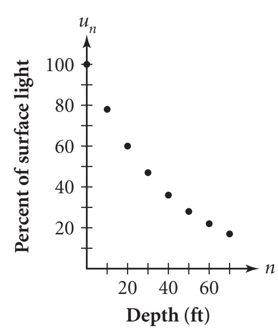
Marine life near the ocean's surface relies on organisms that use the sun for photosynthesis. But lifeforms at the bottom of the ocean, where sunlight is virtually absent, feed on waste material or microorganisms that create energy from chemicals released from the Earth's crust, a process called chemosynthesis. Squat lobsters and galatheid crabs are among many deep sea lifeforms that thrive near hydrothermal vents, where the Earth's crust releases chemical compounds.



▶ Solution

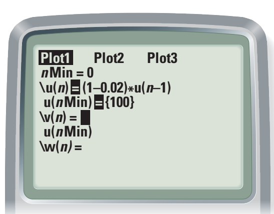
A graph of the data shows a decreasing, curved pattern. It is not linear, so an arithmetic sequence is not a good model. A geometric sequence with a long-run value of 0 will be a better choice.

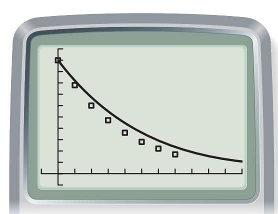
The starting value at depth 0 ft is 100% light intensity, so use $u_0 = 100$. The recursive rule should have the form $u_n = (1 - p)u_{n-1}$, but the data are not given for every foot so you cannot immediately find a common ratio. The ratios between the given values are all approximately 0.77, or $(1 - 0.23)$. Because the light intensity decreases at a rate of 0.23 every 10 feet, it must decrease at a smaller rate every foot. A starting guess of 0.02 gives the model



$$u_0 = 100$$
$$u_n = (1 - 0.02)u_{n-1} \quad \text{where } n \geq 1$$

Check this model by graphing the original data and the sequence on your calculator. The graph shows that this model fits only one data point—it does not decay fast enough. [▶ See **Calculator Note 1D** to learn about sequences on your calculator and **Calculator Note 1E** to learn about graphing sequences. ◀]

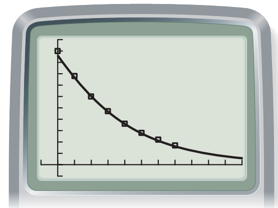




[−10, 110, 10, −10, 110, 10]

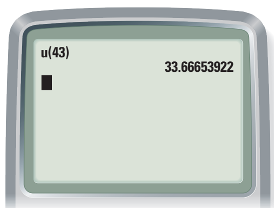
Experiment by increasing the rate of decay. With some trial and error, you can find a model that fits the data better.

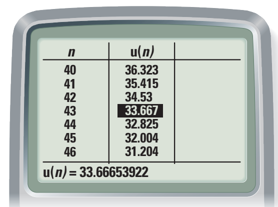
$$u_0 = 100$$
$$u_n = (1 - 0.025)u_{n-1} \quad \text{where } n \geq 1$$



[−10, 110, 10, −10, 110, 10]

Once you have a good sequence model, you can use your calculator to find specific terms or make a table of terms [▶ See **Calculator Note 1K** to learn how to make a table of sequence values. ◀]. For example, the value of u_{43} means that at depth 43 ft approximately 34% of surface light intensity remains.





As you see in the calculator screens in the previous example, some calculators use $u(0)$, $u(1)$, $u(2)$, . . . , $u(n-1)$, and $u(n)$ instead of the subscripted notation u_0, u_1, u_2, . . . , u_{n-1}, and u_n. Be aware that $u(5)$ means u_5, not u multiplied by 5. You may also see other variables, such as a_n or v_n, used for recursive formulas in other textbooks. It is important that you are able to make sense of these equivalent mathematical notations and be flexible in reading other people's work.

Being alert also pays off when working with graphs. Graphs help you understand and explain situations, and visualize the mathematics of a situation. When you make a graph or look at a graph, try to find connections between the graph and the mathematics used to create the graph. Consider what variables and units were used on each axis and what the smallest and largest values were for those variables. Sometimes this will be clear and obvious, and sometimes you will need to look at the graph in a new way to see the connections.

EXERCISES

Practice Your Skills

1. Suppose you are going to graph the specified terms of these four sequences. For each sequence, what minimum and maximum values of n and u_n would you use on the axes to get a good graph?

a.

n	0	1	2	3	4	5	6	7	8	9
u_n	2.5	4	5.5	7	8.5	10	11.5	13	14.5	16

b. The first 20 terms of the sequence generated by

$$u_0 = 400$$
$$u_n = (1 - 0.18)u_{n-1} \quad \text{where } n \geq 1$$

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c. The first 30 terms of the sequence generated by

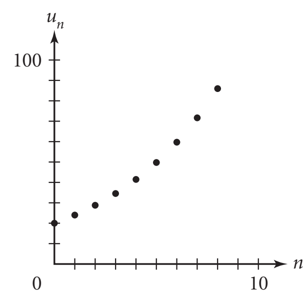
$$u_0 = 25$$
$$u_n = u_{n-1} - 7 \quad \text{where } n \geq 1$$

d. The first 70 terms of the sequence generated by

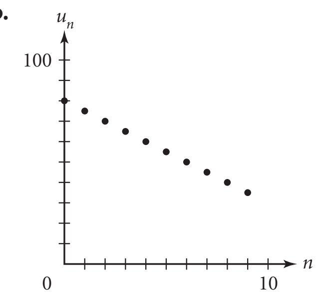
$$u_0 = 15$$
$$u_n = (1 + 0.08)u_{n-1} \quad \text{where } n \geq 1$$

2. Identify each graph as a representation of an arithmetic sequence, a geometric sequence, or a shifted geometric sequence. Use an informed guess to write a recursive formula for each.

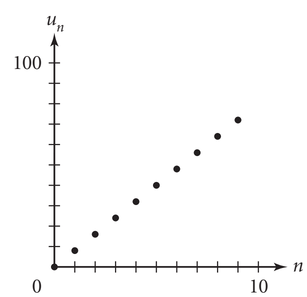
a.



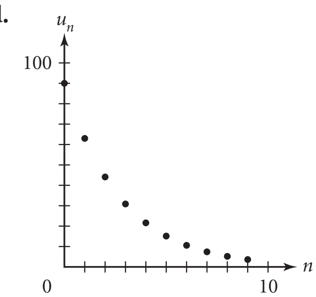
b.



c.



d.



3. Imagine the graphs of the sequences generated by these recursive formulas. Describe each graph using exactly three of these terms: arithmetic, decreasing, geometric, increasing, linear, nonlinear, shifted geometric.

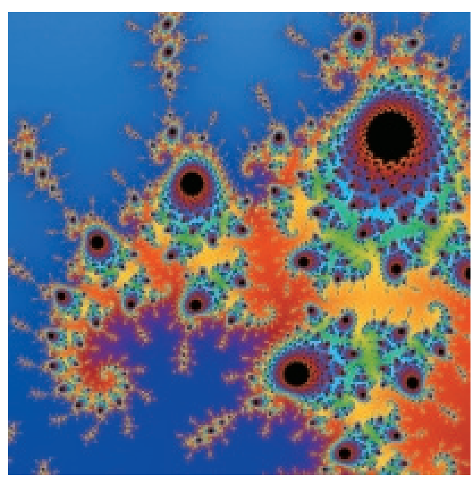
a. $u_0 = 450$
$u_n = a \cdot u_{n-1} \quad \text{where } n \geq 1 \text{ and } 0 < a < 1$

b. $u_0 = 450$
$u_n = b + u_{n-1} \quad \text{where } n \geq 1 \text{ and } b < 0$

c. $u_0 = 450$
$u_n = u_{n-1} \cdot c \quad \text{where } n \geq 1 \text{ and } c > 1$

d. $u_0 = 450$
$u_n = u_{n-1} + d \quad \text{where } n \geq 1 \text{ and } d > 0$

Keymath.com More Practice Your Skills



This complex fractal was created by plotting points generated by recursive formulas.

Reason and Apply

4. Consider the recursive rule $u_n = 0.75u_{n-1} + 210$.

a. What is the long-run value of any shifted geometric sequence that is generated by this recursive rule?

b. Sketch the graph of a sequence that is generated by this recursive rule and has a starting value

i. Below the long-run value
ii. Above the long-run value
iii. At the long-run value

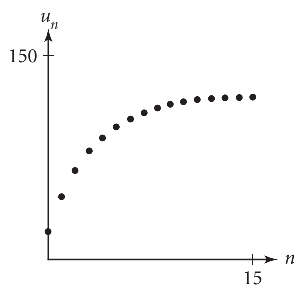
c. Write a short paragraph describing how the long-run value and starting value of each shifted geometric sequence in 4b influence the appearance of the graph.

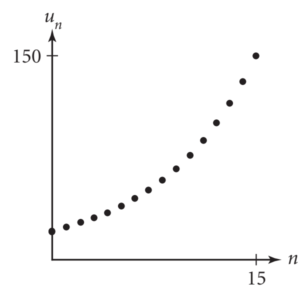
5. Match each recursive formula with the graph of the same sequence. Give your reason for each choice.

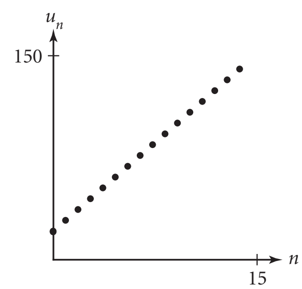
A. $u_0 = 20$
$u_n = u_{n-1} + d$ where $n \geq 1$

B. $u_0 = 20$
$u_n = r \cdot u_{n-1}$ where $n \geq 1$

C. $u_0 = 20$
$u_n = r \cdot u_{n-1} + d$ where $n \geq 1$

i.


ii.


iii.


6. Consider the geometric sequence 18, −13.5, 10.125, −7.59375, ...

a. Write a recursive formula that generates this sequence.

b. Sketch a graph of the sequence. Describe how the graph is similar to other graphs that you have seen and also how it is unique.

c. What is the long-run value?

7. Your friend calls on the phone and the conversation goes like this:

Friend: What does the graph of an arithmetic sequence look like?

You: Be more specific.

Friend: Why? Don't they all look the same?

You: Yes and no.

Explain for your friend how the graphs of arithmetic sequences are similar and how they vary.

8. Your friend calls back and asks about geometric sequences. Explain how the graphs of geometric sequences are similar and how they vary.

9. **APPLICATION** The Forever Green Nursery has 7000 white pine trees. Each year, the nursery plans to sell 12% of its trees and plant 600 new ones.

a. Make a graph that shows the number of trees at the nursery over the next 20 years.

b. Use the graph to estimate the number of trees in the long run. How does your estimate compare to the long-run value you found in Exercise 8b in Lesson 1.3?

10. **APPLICATION** The Bayside Community Water District has decided to add fluoride to the drinking water. Board member Evelyn King does research and finds that the ideal concentration of fluoride in drinking water is between 1.00 mg/L and 2.00 mg/L. If the concentration gets higher than 4.00 mg/L, people may suffer health problems, such as bone disease or damage to developing teeth. If the concentration is less than 1.00 mg/L, it is too low to promote dental health. Ms. King supposes 15% of the fluoride present in the water supply is consumed during a period of one day. Create a graph to help her analyze each of these scenarios.


A water treatment plant

a. If the fluoride content begins at 3.00 mg/L and no additional fluoride is added, how long will it be before the concentration is too low to promote dental health?

b. If the fluoride content begins at 3.00 mg/L and 0.50 mg/L is added daily, will the concentration increase or decrease? What is the long-run value? Explain your reasoning.

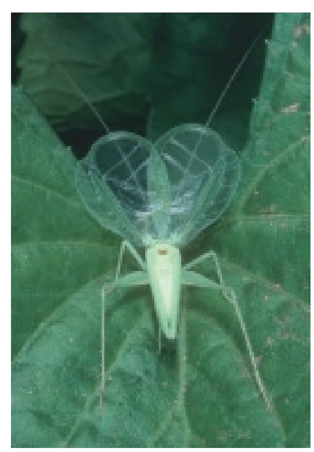
c. Suppose the fluoride content begins at 3.00 mg/L and 0.10 mg/L is added daily. Describe what happens.

d. The Water District board members vote that there should be an initial treatment of 3.00 mg/L, but that the long-run fluoride content should be 1.50 mg/L. How much fluoride needs to be added daily for the fluoride content to stabilize at 1.50 mg/L?

11. **APPLICATION** As the air temperature gets warmer, snowy tree crickets chirp faster. You can actually use a snowy tree cricket's rate of chirping per minute to determine the approximate temperature in degrees Fahrenheit. Use a graph to find a sequence model that approximately fits these data.

Snowy Tree Crickets' Rate of Chirping

Temperature (°F)	50	55	60	65	70	75	80
Rate (chirps/min)	40	60	80	100	120	140	160



Snowy tree crickets are about 0.7 in. long, pale green, and live in shrubs and bushes. Only male crickets chirp, and they have different chirps for different activities, such as mating and fighting. All species of crickets chirp by rubbing their wings together.

12. **APPLICATION** This table gives the estimated population of Peru from 1950 to 2000. Use a graph to find a sequence model that approximately fits these data.

Population of Peru

Year	Population (millions)
1950	7.6
1960	9.9
1970	13.2
1980	17.3
1990	22.0
1996	25.1
2000	27.0

(U.S. Bureau of the Census, International Data Base)

This painting, *Corpus Christi 1982,* by Antonio Huillca Huallpa shows a scene in Cuzco, Peru.

Review

13. Consider the recursive formula

$$u_0 = 450$$
$$u_n = 0.75u_{n-1} + 210 \quad \text{where } n \geq 1$$

a. Find u_1, u_2, u_3, u_4, and u_5.

b. How can you calculate backward from the value of u_1 to u_0, or 450? In general, what operations can you perform to any term in order to find the value of the previous term?

c. Write a recursive formula that generates the values of u_5 to u_0 backward.

14. Find the value of a that makes each equation true.

a. $47{,}500{,}000 = 4.75 \times 10^a$ **b.** $0.0461 = a \times 10^{-2}$ **c.** $3.48 \times 10^{-1} = a$

15. ***Mini-Investigation*** For a–c, find the long-run value of the sequence generated by the recursive formula.

a. $u_0 = 50$
$u_n = (1 - 0.30)u_{n-1} + 10 \quad \text{where } n \geq 1$

b. $u_0 = 50$
$u_n = (1 - 0.30)u_{n-1} + 20 \quad \text{where } n \geq 1$

c. $u_0 = 50$
$u_n = (1 - 0.30)u_{n-1} + 30 \quad \text{where } n \geq 1$

d. Generalize any patterns you notice in your answers to 15a–c. Use your generalizations to find the long-run value of the sequence generated by

$$u_0 = 50$$
$$u_n = (1 - 0.30)u_{n-1} + 70 \quad \text{where } n \geq 1$$

EXPLORATION

Recursion in Geometry

In this chapter you have used recursive rules to produce sequences of numbers. In geometry, you may have used recursive rules to produce fractals or other geometric shapes. In this exploration you will explore two geometric designs that follow recursive rules.

The shell of the chambered nautilus, a relative of the octopus, reflects the Fibonacci sequence. It is only one of many occurrences of Fibonacci numbers in nature.

In Part 1, you will use The Geometer's Sketchpad® to create a Fibonacci spiral. In Part 2, you will construct a golden rectangle spiral, then compare the two constructions. You can learn more about mathematical spirals by visiting the Internet links at www.keymath.com/DAA .

Activity

Two Spirals

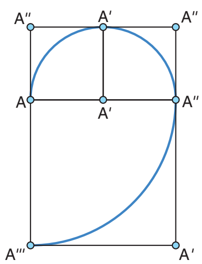
Part 1: The Fibonacci Spiral

In his book *Liber abaci* (1202), Italian mathematician Leonardo Fibonacci (ca. 1170–1240) posed this question:

> How many pairs of rabbits will be produced in a year, beginning with a single pair, if every month each pair bears a new pair which becomes productive one month later?

You can model this scenario with the sequence 1, 1, 2, 3, 5, 8, 13, ...

Can you describe, in words, the rules of this sequence? Can you write a recursive formula that generates this sequence? (*Hint:* In a sequence, u_n need not depend only on u_{n-1}.) Interestingly, the Fibonacci sequence has applications in mathematics and life science. The sequence models such things as honeybee populations, the seed patterns in the middle of a sunflower, and the number of branches on a tree as it grows.



The first three stages of the Fibonacci spiral.

Procedure Note

Creating a Measured Square Tool with an Arc

1. In a new sketch, construct two points, A and B.
2. Rotate point B about point A by 90°. Then, rotate point A about point B' by 90°. Select, in order, A, B, A', and B'. Choose **Segments** from the Construct menu to create square $ABA'B'$.
3. Using the circle tool, construct a circle with point B as its center through point A. Construct the minor arc, $\overset{\frown}{A'A}$, then hide the circle.
4. Measure the distance between points A and B. Deselect all objects. Select, in order, points A and B, then choose **Select All** from the Edit menu.
5. Click and hold the **Custom Tool** icon and choose **Create New Tool . . .** from the pop-up menu. Name the tool Measured Square.

Step 1 Follow the Procedure Note to create the Measured Square tool. Then open a new sketch.

Step 2 Construct point A. Translate point A a distance of 1 cm at 0°.

Step 3 Using the Measured Square tool, select point A then point A'.

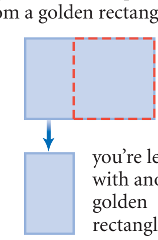
Step 4 Now select the new point A' created by your tool, followed by the original point A'. Next select the new point A'' followed by the original point A. Continue this process, creating larger and larger squares, always choosing the point at the end of the new arc first, followed by the other endpoint of the longest side.

Step 5 The spiral you have created is the Fibonacci spiral. What do you notice about the design?

Step 6 Describe the process that creates the Fibonacci spiral. How does this relate to the Fibonacci sequence?

Part 2: The Golden Rectangle Spiral

If you remove a square from a golden rectangle,



you're left with another golden rectangle.

The **golden ratio** is a number, often represented with the Greek letter ϕ (lowercase phi). One definition of phi is that to square it you just add one to it, or

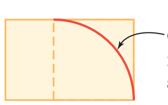
$$\phi^2 = \phi + 1$$

You can rearrange this equation and use the quadratic formula to solve for ϕ.

$$\phi = \frac{1 \pm \sqrt{5}}{2}$$

The positive root, which is approximately 1.618, is the golden ratio.

The **golden rectangle** is a rectangle whose length is ϕ times its width. The golden rectangle spiral is produced by drawing a segment to create a square inside the golden rectangle and constructing a 90° arc to span the corners of that square. A golden rectangle is unique because the smaller rectangle that remains is also a golden rectangle, so the process can be repeated.



Golden rectangle spiral

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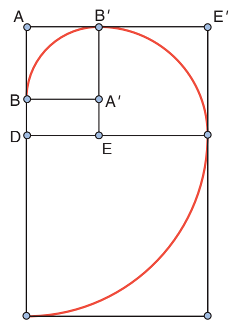


The first three stages of the golden rectangle spiral.

Procedure Note

Golden Rectangle Spiral

1. In a new sketch, construct two points, A and B.
2. Rotate point B around point A by 90°. Then, rotate point A around point B' by 90°. Select, in order, A, B, A', and B'. Choose **Segments** from the Construct menu to create square $ABA'B'$.
3. Construct the midpoint C of side AB. Construct a circle centered at point C through point A'.
4. Construct a ray, $\overrightarrow{AB}$. Mark the intersection of the ray and the circle point D. Hide the circle, the ray, and point C.
5. Construct a rectangle with D, A, and B' as three of its vertices. Label the fourth vertex E. Hide any construction lines.
6. Construct a circle centered at point A' and containing points B' and B. Construct the minor arc, $\overset{\frown}{BB'}$, then hide the circle.
7. Rotate point E about point B' by 90° to create point E'. Select points A and B and choose **Iterate** from the Transform menu. Map point A to point E' and point B to point B'.

Step 1 Use the Procedure Note to create a golden rectangle spiral.

Step 2 Choose **Select All** from the Edit menu. Use the "+" key on your keyboard to add iterations to your sketch.

Step 3 The spiral you have created is the golden rectangle spiral. What is the ratio of length to width in each rectangle?

Step 4 Are all the rectangles in your design similar? Why or why not?

Questions

1. Print out each of your sketches so that you can compare them by laying one over the other. (When printing, you should select the Scale to Fit Page option in the Print Preview dialog box.) Is there a resemblance between the golden rectangle spiral and the Fibonacci spiral? How are they different?
2. Write the sequence of length-to-width ratios for the outermost rectangle at each stage of your golden rectangle spiral. Write the sequence of the length-to-width ratios for the outermost rectangle at each stage of your Fibonacci spiral. How are these sequences related?
3. How is recursion in geometry related to recursion in algebra?

Keymath.com Links to Resources

LESSON

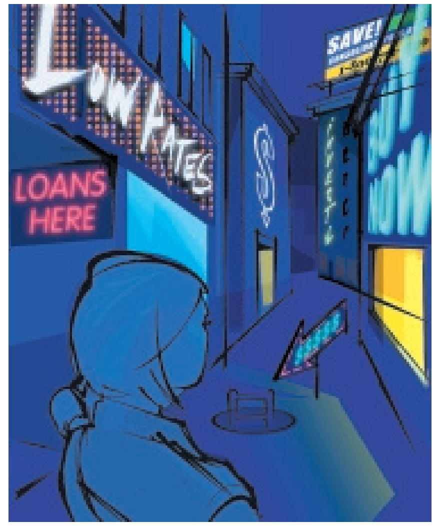
1.5

Loans and Investments

. . . so little stress is ever laid on the pleasure of becoming an educated person, the enormous interest it adds to life.

EDITH HAMILTON

In life you will face many financial situations, which may include car loans, checking accounts, credit cards, long-term investments, life insurance, retirement accounts, and home mortgages. You will need to make intelligent choices about your money and whom you can trust. Fortunately, much of the mathematics is no more complicated than the recursive rule $u_n = r \cdot u_{n-1} + d$.



Investigation

Life's Big Expenditures

In this investigation you will use recursion to explore loan balances and payment options. Your calculator will be a helpful tool for trying different sequence models.

Part 1

You plan to borrow $22,000 to purchase a new car. The investment must be paid off in 5 years (60 months). The bank charges interest at an annual rate of 7.9%, compounded monthly. Part of each monthly payment is applied to the interest, and the remainder reduces the starting balance, or principal.

Step 1 What is the *monthly* interest rate? What is the first month's interest on the $22,000? If you make a payment of $300 at the end of the first month, then what is the remaining balance?

Step 2 Record the balances for the first 6 months with monthly payments of $300. How many months would it take to pay off the loan?

Step 3 Experiment with other values for the monthly payment. What monthly payment allows you to pay off the loan in exactly 60 months?

Step 4 How much do you actually pay for the car using the monthly payment you found in Step 3? (*Hint:* The last payment should be a little less than the other 59 payments.)

Part 2

Use the techniques that you discovered in Part 1 to find the monthly payment for a 30-year home mortgage of $146,000 with an annual interest rate of 7.25%, compounded monthly. How much do you actually pay for the house?

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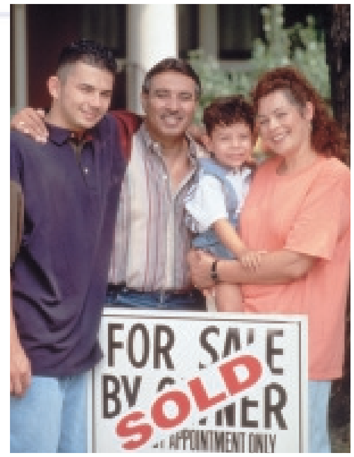
Consumer CONNECTION

In the last quarter of the year 2000, the National Association of Home Builders reported that the median price for a home in the United States was \$151,000. Mortgage lenders usually require monthly payments to be no more than 29% of a family's monthly income, which means that less than 60% of families could afford to buy a home in 2000. The most affordable place to buy a house was Des Moines, Iowa, (median home price of \$107,000) where people with median incomes could afford 88.9% of the homes sold. In contrast, the least affordable houses were in San Francisco, California, where the median home price was \$530,000, and people earning the median income could afford only 6.1% of the homes sold.



Investments are mathematically similar to loans. With an investment, deposits are added on a regular basis so that your balance increases.

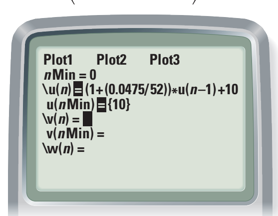
EXAMPLE

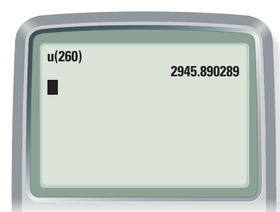
Gwen's employer offers an investment plan that invests a portion of each paycheck before taxes are deducted. Gwen gets paid every week. The plan has a fixed annual interest rate of 4.75%, compounded weekly, and she decides to contribute \$10 each week. What will Gwen's balance be in 5 years?

▸ Solution

Gwen's starting balance is \$10. Each week, the previous balance is multiplied by $\left(1+\frac{0.0475}{52}\right)$ and Gwen adds another \$10. A recursive formula that generates the balance is

$$u_0 = 10$$
$$u_n = \left(1+\frac{0.0475}{52}\right)u_{n-1} + 10 \quad \text{where } n \geq 1$$





There are 52 weeks in a year and 260 weeks in 5 years. The value of u_{260} shows that the balance after 5 years is \$2945.89.

As you work on each exercise, look for these important pieces of information: the principal, the deposit or payment amount, the annual interest rate, and the frequency with which interest is compounded. You will be able to solve many financial problems with these values using recursion.

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EXERCISES

Practice Your Skills

1. Assume the sequence generated by $u_0 = 450$ and $u_n = (1 + 0.039)u_{n-1} + 50$ where $n \geq 1$ represents a financial situation and n is measured in years.

a. Is this a loan or an investment? Explain your reasoning.

b. What is the principal?

c. What is the deposit or payment amount?

d. What is the annual interest rate?

e. What is the frequency with which interest is compounded?

2. Answer the questions in Exercise 1a–e for the sequence generated by $u_0 = 500$ and $u_n = \left(1 + \frac{0.04}{4}\right)u_{n-1} - 25$ where $n \geq 1$. Let n be measured in quarter-years.

3. Find the first month's interest on a \$32,000 loan at an annual interest rate of

a. 4.9% **b.** 5.9% **c.** 6.9% **d.** 7.9%

4. Write a recursive formula for each financial situation.

a. You borrow \$10,000 at an annual interest rate of 10%, compounded monthly, and each payment is \$300.

b. You buy \$7000 worth of furniture on a credit card with an annual interest rate of 18.75%, compounded monthly. You plan to pay \$250 each month.

c. You invest \$8000 at 6%, compounded quarterly, and you deposit \$500 every three months. (Quarterly means four times per year.)

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Your Skills

d. You enroll in an investment plan that deducts \$100 from your monthly paycheck and deposits it into an account with an annual interest rate of 7%, compounded monthly.

Reason and Apply

5. **APPLICATION** Find the balance after 5 years if \$500 is deposited into an account with an annual interest rate of 3.25%, compounded monthly.

6. **APPLICATION** Consider a \$1000 investment at an annual interest rate of 6.5%, compounded quarterly. Find the balance after

a. 10 years **b.** 20 years **c.** 30 years

7. ***Mini-Investigation*** Find the balance of a \$1000 investment, after 10 years, at an annual interest rate of 6.5% when compounded

a. Annually

b. Monthly

c. Daily (In financial practice, daily means 360 times per year, not 365.)

d. After the same length of time, how will the balances compare from investments that are compounded annually, monthly, or daily?

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8. **APPLICATION** Beau and Shaleah each get a $1000 bonus at work and decide to invest it. Beau puts his money into an account that earns an annual interest rate of 6.5%, compounded yearly. He also decides to deposit $1200 each year. Shaleah finds an account that earns 6.5%, compounded monthly, and decides to deposit $100 each month.

a. Compare the amounts of money that Beau and Shaleah deposit each year. Describe any differences or similarities.

b. Compare the balances of Beau's and Shaleah's accounts over several years. Describe any differences or similarities.

9. **APPLICATION** Regis deposits $5000 into an account for his 10-year-old child. The account has an annual interest rate of 8.5%, compounded monthly.

a. What regular monthly deposit amount is needed to make the account worth $1 million by the time the child is 55 years old?

b. Make a graph of the increasing balances.

10. **APPLICATION** Cici purchased $2000 worth of merchandise with her credit card this past month. Then she was unexpectedly laid off from her job. She decided to make no more purchases with the card and to make only the minimum payment of $40 each month. Her annual interest rate is 18%, compounded monthly.

a. Find the balance on the credit card over the next six months.

b. When will Cici pay off the total balance on her credit card?

c. What is the total amount paid for the $2000 worth of merchandise?

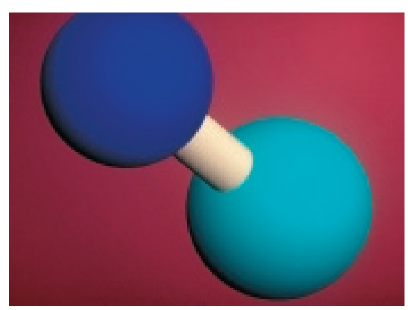
11. **APPLICATION** Megan Flanigan is a loan officer with L. B. Mortgage Company. She offers a loan of $60,000 to a borrower at 9.6% annual interest, compounded monthly.

a. What should she tell the borrower the monthly payment will be if the loan must be paid off in 25 years?

b. Make a graph that shows the unpaid balance over time.

Review

12. A mixture of nitric oxide (NO, a colorless gas) and dinitric oxide (N_2O_2, a red-brown gas) exists in equilibrium. A mixture of 20 mL of NO and 220 mL of N_2O_2 is heated. At the new temperature, 10% of the NO changes to N_2O_2 each second and 5% of the N_2O_2 changes to NO each second.



A model of a molecule of nitric oxide

a. Calculate the amount of NO that was changed into N_2O_2 during the first second.

b. Calculate the amount of N_2O_2 that was changed into NO during the first second.

c. Assume the container is sealed and will always contain 240 mL of gas. From your answers to 12a and b, find the amount of NO after 1 second.

d. How much of the gas is N_2O_2 after 1 second?

e. What will be the amounts of NO and N_2O_2 after 3 seconds? 10 seconds? What will happen in the long run?

13. Is this sequence arithmetic, geometric, shifted geometric, or something else?

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

14. Consider the geometric sequence generated by

$$u_0 = 4$$
$$u_n = 0.7u_{n-1} \quad \text{where } n \geq 1$$

a. What is the long-run value?

b. What is the long-run value if the common ratio is changed to 1.3?

c. What is the long-run value if the common ratio is changed to 1?

15. Find the value of n in each proportion.

a. $\dfrac{2.54\text{ cm}}{1\text{ in.}} = \dfrac{n}{12\text{ in.}}$ **b.** $\dfrac{1\text{ km}}{0.625\text{ mi}} = \dfrac{n}{200\text{ mi}}$ **c.** $\dfrac{1\text{ yd}}{0.926\text{ m}} = \dfrac{140\text{ yd}}{n}$

Project

THE PYRAMID INVESTMENT PLAN

Have you ever received a chain letter offering you prizes or great riches? The letters ask you to follow the simple instructions and not break the chain. Actually, chain-letter schemes are illegal, even though they continue to be quite common.

Suppose you have just received this letter, along with several quotes from "ordinary people" who have already become millionaires. How many rounds have already taken place? How many more rounds have to take place before you become a millionaire?

Your project should include

- A written analysis of this plan.
- Your conclusion about whether or not you will become a millionaire.
- An answer to the question, "Why do you think chain letter schemes and pyramid plans are illegal?"

Join the Pyramid Investment Plan (PIP)!

Become a millionaire! Send only $20 now, and return it with this letter to PIP. PIP will send $5 to the name at the top of the list below. Then the second person will move up to the top of the list, and your name will be added to the bottom of the list. You will receive a new letter and a set of 200 names and addresses. Your name will be in position #6. Make 200 copies of the letter, mail them, and **wait to get rich!**

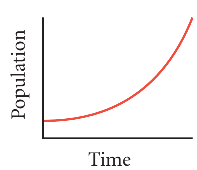
Each time a recipient of one of your letters joins PIP, your name advances toward the top of the list. When your name reaches the top, each of the thousands of people who receive that letter will be sending money to PIP, and you will receive your share. **A conservative marketing return of 6% projects that you will earn over $1.2 million!**

1. *Chris*
2. *Katie*
3. *Josh*
4. *Kanako*
5. *Dave*
6. *Miranda*

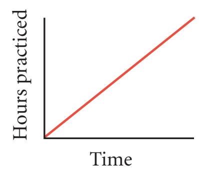
EXPLORATION

Refining the Growth Model

Until now you have assumed that the rate of growth or decay remains constant over time. The terms generated by the recursive formulas you have used to model arithmetic or geometric growth increase infinitely in the long run.



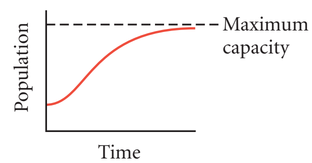
Geometric growth



Arithmetic growth

However, environmental factors rarely support unlimited growth. Because of space and resource limitations, or competition among individuals or species, an environment usually supports a population only up to a limiting value.

The graph at right shows a **logistic function** with the population leveling off at the maximum capacity. If a population's growth is modeled by a logistic function, the growth rate isn't a constant value, but rather changes as the population changes.



Logistic functions are used to model more than just population. Researchers in many fields apply logistic functions to study things such as the spread of disease or consumer buying patterns. In this exploration you will see an example of logistic growth.

Environmental CONNECTION

Some desert cities in the United States have growing populations and little supply of natural resources. Large populations need, for example, a plentiful water supply for drinking and sewage systems, as well as for luxuries such as watering lawns and filling swimming pools. Population growth reduces groundwater supply, which in turn activates old earthquake faults and surface fissures, and damages buildings. Using logistic functions to model population growth can help government agencies monitor natural resources and avoid environmental catastrophes.

The Hoover Dam, located on the border of Nevada and Arizona, helps supply water to desert cities such as Las Vegas.

Activity

Cornering the Market

A company has invented a new gadget, and everyone who hears about it wants one! The company hasn't started advertising the new gadget yet, but the news is spreading fast. Simulate this situation with your class to see what happens when a popular new product enters the market.

Each person in your class will be assigned an I.D. number. At time zero, only one person has bought the gadget, and at the end of every time period, each person who has one tells another person about it, and they go out and buy one (unless they already have one).

Step 1 For about 10 time periods each person who knows about the gadget generates a random I.D. number and tells that person, who immediately goes out and buys one. [▶ See **Calculator Note 1L** to learn how to generate random numbers. ◀] Create a table like the one at right, and record the total number of people who have the gadget.

Time period	Number of people who own the gadget
0	1
1	
2	
3	

Step 2 Enter your data into lists L1 and L2. Make a scatter plot of your data. Describe your scatter plot. Explain why the number of people who own the gadget doesn't always double for each time period.

Step 3 Divide each term in list L2 by the previous term and enter these ratios in list L3. These ratios show you the rate at which the number of people who own the gadget grows during each time period.

Step 4 In this activity the growth rate depends on the number of people who own the gadget. Shorten list L1 and list L2 by deleting the last value in each list so that all three lists are the same length. Turn off the scatter plot from Step 2 and make a new scatter plot of (L2, L3). What happens to the growth rate as the number of people who own the gadget increases?

The net growth at each step depends on the previous population size u_{n-1}. So the net growth is a function of the population. This changes the simple growth model of $u_n = u_{n-1}(1 + p)$, in which the rate, p, is a constant, to a growth model in which the growth rate changes depending upon the population.

$$u_n = u_{n-1}\left(1 + p \cdot \left(1 - \frac{u_{n-1}}{L}\right)\right)$$

where p is the unrestricted growth rate, L is the limiting capacity or maximum population, and $p \cdot \left(1 - \frac{u_{n-1}}{L}\right)$ is the net growth rate.

Step 5 What is the maximum population, L, for the gadget-buying scenario? What is u_0? What is the unrestricted growth rate, p? What does $\frac{u_{n-1}}{L}$ represent? Write the recursive formula for this logistic function.

Step 6 Create list L4, defined as $1 + p \cdot \left(1 - \frac{\text{L2}}{L}\right)$, using p and L from Step 5. Plot (L2, L4) and (L2, L3).

Step 7 Turn off the scatter plots from Step 6 and check how well your recursive formula from Step 5 models your original data (L1, L2).

History CONNECTION

At the end of the 18th century, data on population growth were not available, and social scientists generally agreed with English economist Thomas Malthus (1766–1834) that a population always increases exponentially, eventually leading to a catastrophic overpopulation. In the 19th century, Belgian mathematician Pierre François Verhulst (1804–1849) and Belgian social statistician Adolphe Quételet (1796–1874) formulated the net growth rate expression $p \cdot \left(1 - \frac{u_{n-1}}{L}\right)$ for a population model. Quételet believed that limitations on population growth needed to be accounted for in a more systematic manner than Malthus described. Verhulst was able to incorporate the changes in growth rate in a mathematical model.

A "closed" environment creates clear limitations on space and resources and calls for a logistic function model.

EXAMPLE

Suppose the unrestricted growth rate of a deer population on a small island is 12% annually, but the island's maximum capacity is 2000 deer. The current deer population is 300.

a. What net growth rate can you expect for next year?

b. What will the deer population be in one year?

c. Graph the deer population over the next 50 years.

▶ Solution

a. There is a maximum capacity, so the population can be modeled with a logistic function. The net growth rate is $p\left(1 - \frac{u_{n-1}}{L}\right)$, where p is the unrestricted growth rate and L is the limiting value, in this case, the island's maximum capacity. Because $p = 12\%$ and $L = 2000$ deer, the net growth rate will be

$$0.12\left(1 - \frac{u_{n-1}}{2000}\right)$$

Using $u_0 = 300$, this gives a growth rate of $0.12\left(1 - \frac{300}{2000}\right) = 0.102$, or 10.2%, for the first year.

b. The recursive rule for this logistic function

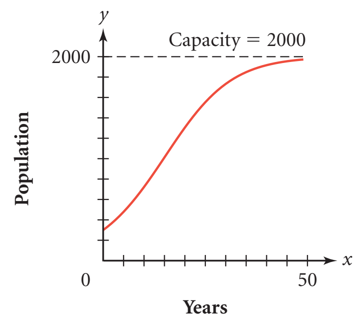
is $u_n = u_{n-1}\left(1 + 0.12\left(1 - \frac{u_{n-1}}{2000}\right)\right)$

Using $u_0 = 300$,

$$u_1 = 300\left(1 + 0.12\left(1 - \frac{300}{2000}\right)\right)$$
$$= 300(1 + 0.102)$$
$$= 330.6$$

So the deer population after 1 year will be around 331.

c. The graph shows the population as it grows toward the maximum capacity of 2000 deer.



Now try your hand at applying logistic function models with these questions.

Questions

These eastern gray kangaroos mingle with tourists on Pebbly Beach in Murramarang National Park, Australia. When humans inhabit areas previously dominated by animals, population growth may be adversely affected.

1. Bacteria grown in a culture dish are provided with plenty of food but a limited amount of growing space. Eventually the population will become overcrowded, even though there is plenty of food. The bacteria grow at an unrestricted rate of 125% per week initially. The starting population is 50, and the capacity of the dish is 5000. Find the net growth rate and the population at the end of each week for 7 weeks.

2. A large field provides enough food to feed 500 healthy rabbits. When food and space are unlimited, the population growth rate of the rabbits is 20%, or 0.20. The population that can be supported is 500 rabbits. Complete each statement using the choices less than 0, greater than 0, 0, or close to 0.20.

a. When the population is less than 500, the net growth rate will be __?__.

b. When the population is more than 500, the net growth rate will be __?__.

c. When the population is very small, the net growth rate will be __?__.

d. When the population is 500, the net growth rate will be __?__.

3. Suppose the recursive rule $d_n = d_{n-1}\left(1 + 0.35\left(1 - \frac{d_{n-1}}{750}\right)\right)$ will give the number of daisies growing in the median strip of a highway each year. Presently there are about 100 daisies. Write a paragraph or two explaining what will happen. Explain and support your reasoning.

CHAPTER 1 REVIEW

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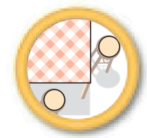
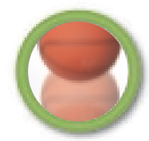


A **sequence** is an ordered list of numbers. In this chapter you used **recursion** to define sequences. A **recursive formula** specifies one or more starting terms and a **recursive rule** that generates the nth term by using the previous term or terms. You learned to calculate the terms of a sequence by hand and by using recursion and sequences on your calculator.



There are two special types of sequences—arithmetic and geometric. **Arithmetic sequences** are generated by always adding the same number, called the **common difference,** to get the next term. Your salary for a job on which you are paid by the hour is modeled by an arithmetic sequence. **Geometric sequences** are generated by always multiplying by the same number, called the **common ratio,** to get the next term. The growth of money in a savings account is modeled by a geometric sequence. For some **growth** and **decay** scenarios, it helps to write the common ratio as a percent change, $(1 + p)$ or $(1 - p)$. Some real-world situations, such as medicine levels, are modeled by **shifted geometric sequences** that use a recursive rule with both multiplication and addition.

Many sequences have a long-run value after many, many terms. Looking at a graph of the sequence may help you see the long-run value. Graphs also help you recognize whether the data are best modeled by an arithmetic or geometric sequence. The graph of an arithmetic sequence is **linear** whereas the graph of a geometric sequence is curved.

EXERCISES

1. Consider this sequence:

256, 192, 144, 108, . . .

a. Is this sequence arithmetic or geometric?

b. Write a recursive formula that generates the sequence. Use u_1 for the starting term.

c. What is the 8th term?

d. Which term is the first to have a value less than 20?

e. Find u_{17}.

2. Consider this sequence:

3, 7, 11, 15, . . .

a. Is this sequence arithmetic or geometric?

b. Write a recursive formula that generates the sequence. Use u_1 for the starting term.

c. What is the 128th term?

d. Which term has the value 159?

e. Find u_{20}.

3. List the first five terms of each sequence. For each set of terms, what minimum and maximum values of n and u_n would you use on the axes to make a good graph?

a. $u_1 = -3$
$u_n = u_{n-1} + 1.5$ where $n \geq 2$

b. $u_1 = 2$
$u_n = 3u_{n-1} - 2$ where $n \geq 2$

4. **APPLICATION** Atmospheric pressure is 14.7 pounds per square inch (lb/in.^{2}) at sea level. An increase in altitude of 1 mi produces a 20% decrease in the atmospheric pressure. Mountain climbers use this relationship to determine whether or not they can safely climb a mountain and to periodically calculate their altitude after they begin climbing.

a. Write a recursive formula that generates a sequence that represents the atmospheric pressure at different altitudes.

b. Sketch a graph that shows the relationship between altitude and atmospheric pressure.

c. What is the atmospheric pressure when the altitude is 7 mi?

d. At what altitude does the atmospheric pressure drop below 1.5 lb/in.^{2}?


These mountaineers climbed to an altitude of 11,522 feet to reach the top of Mt. Clark in Yosemite National Park, California.

Environmental CONNECTION

Humans of any age or physical condition can become ill when they experience extreme changes in atmospheric pressure in a short span of time. Atmospheric pressure changes the amount of oxygen a person is able to inhale which, in turn, causes the buildup of fluid in the lungs or brain. Someone who travels from a low-altitude city like Akron, Ohio, to a high-altitude city like Aspen, Colorado, and immediately ascends a mountain to ski, could get a headache, become nauseated, or even become seriously ill.

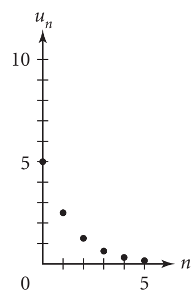
5. Match each recursive formula with the graph of the same sequence. Give your reason for each choice.

A. $u_0 = 5$
$u_n = u_{n-1} + 1$ where $n \geq 1$

B. $u_0 = 1$
$u_n = (1 + 0.5)u_{n-1}$ where $n \geq 1$

C. $u_0 = 5$
$u_n = (1 - 0.5)u_{n-1}$ where $n \geq 1$

D. $u_0 = 5$
$u_n = u_{n-1} - 1$ where $n \geq 1$

i.


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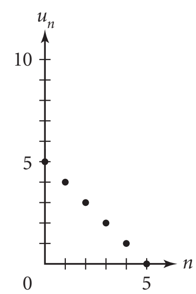

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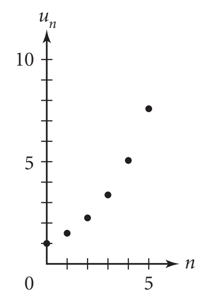

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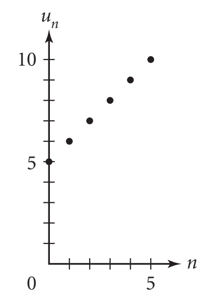
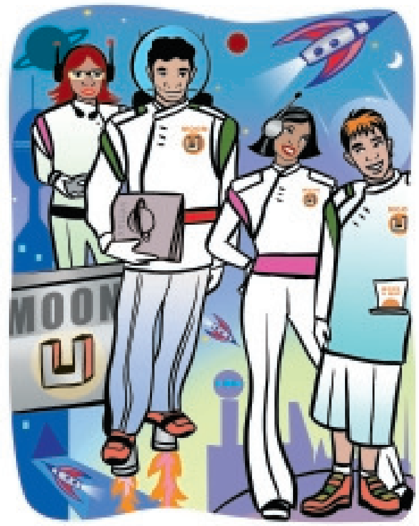


6. A large barrel contains 12.4 gal of oil 18 min after its drain is opened. How many gallons of oil were in the barrel to start, if it drains at a rate of 4.2 gal/min?

7. **APPLICATION** The enrollment at a college is currently 5678. From now on, the board of administrators estimates that each year the school will graduate 24% of its students and admit 1250 new students. What will the enrollment be during the sixth year? What will the enrollment be in the long run? Sketch a graph of the enrollment over 15 years.



8. **APPLICATION** You deposit \$500 into a bank account that has an annual interest rate of 5.5%, compounded quarterly.

a. How much money will you have after 5 yr if you never deposit more money?

b. How much money will you have after 5 yr if you deposit an additional \$150 every 3 mo after the initial \$500?

9. **APPLICATION** This table gives the consumer price index for medical care from 1970 to 2000. Use a graph to find a sequence model that approximately fits these data.

U.S. Consumer Price Index for Medical Care

Year	Consumer price index
1970	34.0
1975	47.5
1980	74.9
1985	113.5
1990	162.8
1995	220.5
2000	260.8

(U.S. Department of Labor, Bureau of Labor Statistics)

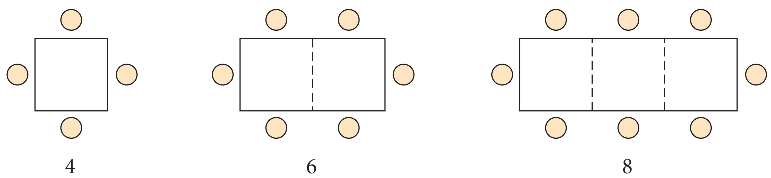
Economics CONNECTION

The consumer price index (CPI) is a measure of the change over time in the prices paid by consumers for goods and services, such as food, clothing, and health care. The Bureau of Labor Statistics obtains price information for 80,000 items in order to adjust the index. The price of specific items in 1982–1984 is assigned an index of 100 and the index for all subsequent years is given in relation to this reference period. For example, an index of 130 means the price of an item increased 30% from the price during 1982–1984. Learn more about the CPI with the links at www.keymath.com/DAA .

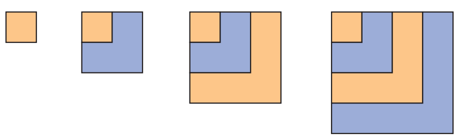
10. **APPLICATION** Oliver wants to buy a cabin and needs to borrow \$80,000. What monthly payment is necessary to pay off the mortgage in 30 years if the annual interest rate is 8.9%, compounded monthly?

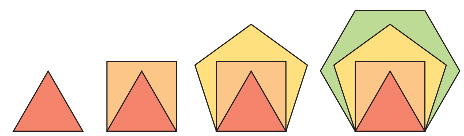
TAKE ANOTHER LOOK

1. In Lesson 1.1, Example A, you saw an arithmetic sequence that generates from a geometric pattern: the number of seats around a linear arrangement of square tables is 4, 6, 8, ... Notice that the sequence comes from the increase in the perimeter of the arrangement. What sequence comes from the increase in area?



Use these geometric patterns to generate recursive sequences using perimeter, area, height, or other attributes, and give their recursive formulas. Which sequences exhibit arithmetic growth? Geometric growth? Are there sequences you cannot define as arithmetic or geometric?





Invent a geometric pattern of your own. Look for and define different sequences.

2. Here's a strategy an algebra student discovered for finding the monthly payment you would need to make to pay off a loan. First, specify the loan amount, the annual interest rate, and the term (length) of the loan. Choose trial monthly payment amounts that form a sequence, say, $0, $1, $2, $3. Record the final balance remaining at the end of the term for each payment. Next, explore the differences in the final balances and find a pattern. Finally, use the pattern to find a monthly payment that results in a zero final balance. Tell how you know that your payment amount is correct. How many trial payment amounts do you need in order to determine the monthly payment? Try this strategy for one of the exercises in Lesson 1.5.

3. Imagine a target for a dart game that consists of a bull's-eye and three additional circles. If it is certain that a dart will land within the target but is otherwise random, what sequence of radii gives a set of probabilities that form an arithmetic sequence? A geometric sequence? Sketch what the targets look like.

4. As you probably realized in Lesson 1.5, your graphing calculator is a useful tool for solving loan and investment problems. With the added help of a special program, your calculator will instantly tell you information such as the monthly payment necessary for a home mortgage. [▶ See **Calculator Note 1M** to learn how to use your calculator's financial-solver program. ◀] Try using this program to solve some of the exercises in Lesson 1.5 or a problem of your own design.

Assessing What You've Learned

ORGANIZE YOUR NOTEBOOK Wouldn't you like to have a record of what you've learned and just what you are expected to know in this algebra course? Your own notebook can be that record if you enter significant information and examples into it on a regular basis. It is a good place for new vocabulary, definitions and distinctions, and worked-out examples that illustrate mathematics that's new to you. On the other hand, if it is simply a stack of returned homework, undated class notes, and scratch-paper computations and graphs, it is too disorganized to perform that service for you.

Before you get very far along in the course, take time to go through your notebook. Here are some suggestions:

- Put papers into chronological order. If your work is undated, use the table of contents in this book to help you reconstruct the sequence in which you produced the items in your notebook. You can number pages to help keep them in order.
- Go through the book pages of Chapter 0 and Chapter 1, and see whether you have a good record of how you spent class time—what you learned from investigations and homework. Fill in notes where you need them while information is fresh in your mind. Circle questions that you need to have cleared up before going on.
- Reflect on the main ideas of this chapter. Organize your notes by type of sequence, and make sure you have examples of recursive formulas that produce different types of graphs. Be sure you can identify the starting value and the common difference or common ratio for each type of sequence, and that you know how to use the information to find a later term in the sequence.

PERFORMANCE ASSESSMENT It's important to know how well you are progressing in this course so that you're encouraged by your gains. Assessing what you've learned also alerts you to get help before you're confused about important ideas that you'll have to build on later in the course.

One way to tell whether or not you understand something is to try to explain it to someone else. Choose one of the modeling problems in Lesson 1.4, such as the population of Peru in Exercise 12, and present it orally to a classmate, a relative, or your teacher. Here are some steps to follow:

- Explain the problem context, and give the listener some background on what the problem calls for mathematically.
- Be sure the listener knows what you mean by finding a model. Then describe how you find the model, using the given information, and how you express mathematically what the model is.
- Comment on how well the model fits the original data.
- Show the listener how to use the model. For example, ask a question about the future population of Peru, and show the listener how to use the model to answer the question. Show at least one way to check that your result is reasonable. Don't forget to interpret the mathematical result in terms of the problem's real-world context.